Unit 1- Set Theory

Lesson 1- The Basics of Set Theory

ets: It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets understand relationships between groups, and to analyze survey data.

- An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.
- A **set** is a collection of distinct objects, called **elements** of the set
- A set can be defined by describing the contents, or by listing the elements of the set, enclosed in {curly brackets}.

EX: Some examples of sets defined by describing the contents:

- The set of all even numbers
- The set of all books written about travel to Chile

EX: Some examples of sets defined by listing the elements of the set:

- {1, 3, 9, 12}
- {red, orange, yellow, green, blue, indigo, purple}

IMPORTANT: A set simply specifies the contents; ***order is not important***.

- The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}.

Notation: Commonly, we will use a variable to represent a set, to make it easier to refer to that set later.

- The symbol $\in$ means "is an element of".

Empty Set: A set that contains no elements, { }, is called the **empty set** and is notated $\emptyset$

EX: Let $A = \{1, 2, 3, 4\}$

- To notate that 2 is element of the set, we'd write $2 \in A$

Subset: A **subset** of a set *A* is another set that contains only elements from the set *A*, but may also contain all the elements of *A*.

- If *B* is a subset of *A*, we write $B \subseteq A$ *(subset OR equal)*

Proper Subset: A **proper subset** is a subset that is not identical to the original set – it contains fewer elements.

- If *B* is a proper subset of *A*, we write $B \subset A$ *(subset but NOT equal)*

EX: Consider these three sets

A = the set of all even numbers $B = \{2, 4, 6\}$ $C = \{2, 3, 4, 6\}$

- Here $B \subset A$ since every element of *B* is also an even number, so is an element of *A*.
    - More formally, we could say $B \subset A$ since if $x \in B$, then $x \in A$.
- It is also true that $B \subset C$.
- *C* is not a subset of *A*, since C contains an element, 3, that is not contained in *A*

EX: The set $A = \{1, 3, 5\}$. What is a larger set this might be a subset of?

$Z = \{1,3,5,6\}$ * many different options *

IMPORTANT

Union: The **union** of two sets contains all the elements contained in either set (or both sets). ★★

- The union is notated $A \cup B$. **(A or B)**
- More formally, $x \in A \cup B$ if $x \in A$ or $x \in B$ (or both) → Look for the keyword "or"

Intersection: The **intersection** of two sets contains only the elements that are in both sets.

- The intersection is notated $A \cap B$. **(A and B)**
- More formally, $x \in A \cap B$ if $x \in A$ and $x \in B$ → Look for the keyword "and"

Complement: The **complement** of a set *A* contains everything that is *not* in the set *A*.

- The complement is notated A', or A^c, or sometimes $\sim A$.

Disjoint: When two sets are **disjoint** it is because they have no elements in common.

EX: Consider the sets:

***A* = {red, green, blue}** ***B* = {red, yellow, orange}** ***C* = {red, orange, yellow, green, blue, purple}**

a) Find $A \cup B$

- The union contains all the elements in either set: $A \cup B$ = {red, green, blue, yellow, orange}
    - *Notice we only list red once.*

b) Find $A \cap B$

- The intersection contains all the elements in both sets: $A \cap B$ = {red}

c) Find $A' \cap C$

- Here we're looking for all the elements that are *not* in set *A* and are also in *C*.
- $A' \cap C$ = {orange, yellow, purple}

EX: Using the sets from the previous example, find $A \cup C$ and $B' \cap A$

$A \cup C$ = elements in A or C → {red, green, blue, orange, yellow, purple} * only list elements once.

$B' \cap A$ = elements not in B and in A → {green, blue}

Notice that in the example above, it would be hard to just ask for A', since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Universal Set: A **universal set** is a set that contains all the elements we are interested in. This would have to be defined by the context.

- A complement is relative to the universal set, so A' contains all the elements in the universal set that are not in *A*.

EX:

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.

- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers

X: Suppose the universal set is U = all whole numbers from 1 to 9.

- If $A = \{1, 2, 4\}$, then $A' = \{3, 5, 6, 7, 8, 9\}$.

As we saw earlier with the expression $A' \cap C$, set operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an **order of operations**.

BEDMAS

EX: Suppose

H = {cat, dog, rabbit, mouse}, F = {dog, cow, duck, pig, rabbit}, W = {duck, rabbit, deer, frog, mouse}

a) Find $(H \cap F) \cup W$

- We start with the intersection: $H \cap F$ = {dog, rabbit}
- Now we union that result with W: $(H \cap F) \cup W$ = {dog, duck, rabbit, deer, frog, mouse}

b) Find $H \cap (F \cup W)$

- We start with the union: $F \cup W$ = {dog, cow, rabbit, duck, pig, deer, frog, mouse}
- Now we intersect that result with H: $H \cap (F \cup W)$ = {dog, rabbit, mouse}

c) Find $(H \cap F)' \cap W$

- We start with the intersection: $H \cap F$ = {dog, rabbit}
- Now we want to find the elements of W that are *not* in $H \cap F$
- $(H \cap F)' \cap W$ = {duck, deer, frog, mouse}

EX: Suppose

$A = \{1, 2, 4, 8\}$ $B = \{2, 3, 5, 7\}$ $C = \{2, 4, 6, 8\}$ universal set! $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

List the elements in the following sets:

$A \cap B$ $\{2\}$

$A \cup B$ $\{1,2,3,4,5,7,8\}$

$A \cap C$ $\{2,4,8\}$

$A \cup C$ $\{1,2,4,6,8\}$

$B \cap C$ $\{2\}$

$B \cup C$ $\{2,3,4,5,6,7,8\}$

$A \cup B \cup C$ $\{1,2,3,4,5,6,7,8\}$

$A \cap B \cap C$ $\{2\}$

A' $\{3,5,6,7,9,10\}$

B' $\{1,4,6,8,9,10\}$

C' $\{1,3,5,7,9,10\}$

$A' \cup B'$ $\{1,3,4,5,6,7,8,9,10\}$

$(A \cap B)'$ $\{1,3,4,5,6,7,8,9,10\}$

$(A \cup B)'$ $\{6,9,10\}$

$(A \cap B) \cup C$ $\{2,4,6,8\}$

$(A \cup C) \cap B'$ $\{1,4,6,8\}$

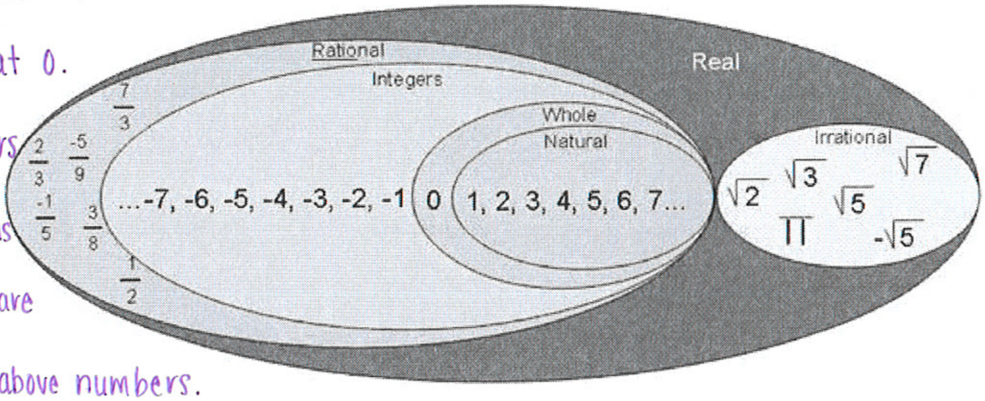
Create another set, labelled D, and ensure it is a disjoint set to A, B, and C.

$D = \{9,10\}$

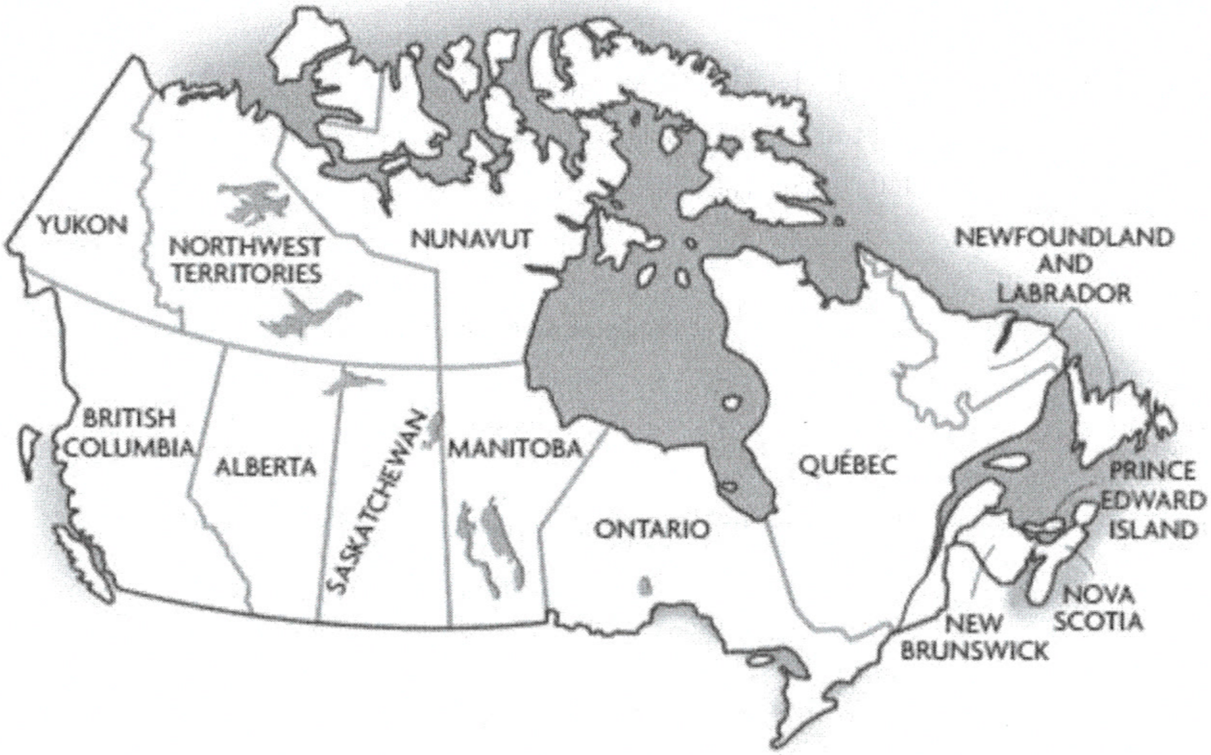
NOTE: You will be required to know the meaning of the following words:

- Sum : adding
- Difference : subtracting
- Product : multiplying
- Quotient : dividing
- Multiple : skip counting → 2: 2,4,6,8...
- Factor : what multiplies to get you that number → 12
  - 1 × 12
  - 2 × 6
  - 3 × 4
  - } factors of 12
- Prime : when a number only has the factor of 1 × itself → 3 (1×3), 7 (1×7)
- Root : x intercepts
- Square : $2^2 = 4$, $3^2 = 9$
- Consecutive : one after another → 1,2,3...
- Odd : 1,3,5,7...
- Even : 0,2,4,6...
- Natural Numbers : starts at 1.
- Whole Numbers : starts at 0.
- Integers : ± whole numbers
- Rationals Numbers : fractions
- Irrational Numbers : pi, square roots
- Real Numbers : all of the above numbers.

Real Number System


Rational
Integers
Whole
Natural
Real
Irrational
7/3
2/3
-5/9
-1/5
3/8
1/2
...-7, -6, -5, -4, -3, -2, -1
0
1, 2, 3, 4, 5, 6, 7...
√2
√3
√7
√5
π
-√5

EX: Use the following picture for the next set of questions:


YUKON
NORTHWEST TERRITORIES
NUNAVUT
NEWFOUNDLAND AND LABRADOR
BRITISH COLUMBIA
ALBERTA
SASKATCHEWAN
MANITOBA
ONTARIO
QUÉBEC
PRINCE EDWARD ISLAND
NOVA SCOTIA
NEW BRUNSWICK

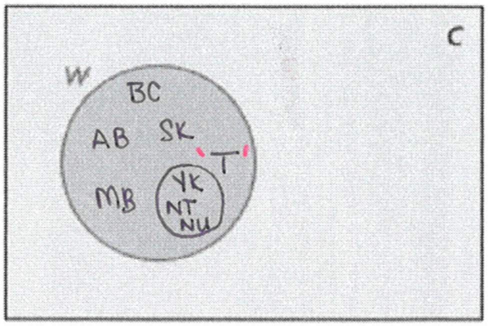
1. List the elements of the universal set of Canadian provinces and territories, C.

$C = \{YK, NT, NU, BC, AB, SK, MB, ON, QC, NL, PE, NS, NB\}$

2. One subset of C is the set of Western Provinces and territories, W. Write W in set notation.

$W = \{YK, NT, NU, BC, AB, SK, MB\}$

3. The Venn diagram below represents the universal set, C. The circle in the Venn Diagram represents the subset W. The complement of W is the set W'.


C
W
BC
AB
SK
MB
'T'
YK
NT
NU

   a. Describe what W' contains

   Not Western provinces + territories

   Eastern provinces + territories!

   b. Write W' in set notation

   $W' = \{ON, QC, NL, PE, NS, NB\}$

   c. Explain what W' represents in the Venn diagram

   Elements in C, but not in W.

4. The set of Eastern provinces are as follows:

$$E = \{NL, PE, NS, NB, QC, ON\}$$

Is E equal to W'? Explain.

Yes! It contains all the same elements. *ORDER does NOT matter*

5. List T, the set of territories in Canada. Is T a subset of C? Is it a subset of W or of W'? Explain using your Venn Diagram.

$T = \{YK, NT, NU\}$, $T \subset C$!

$T \subset W$ because T would be a circle inside of W.

6. Explain why you can represent the set of Canadian provinces south of Mexico by the empty set.

Because there are No provinces south of mexico.

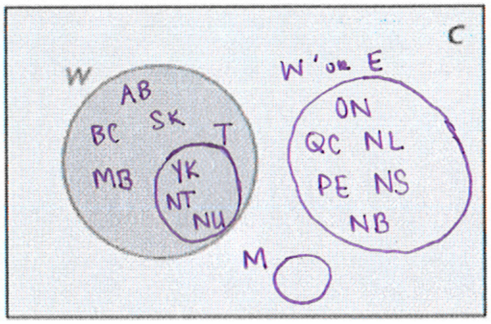
$M = \{\ \}$

7. Consider the set C, W, W' and T. List a pair of disjoint sets. Is there more then one pair of disjoint sets?

W is disjoint to W'

T is disjoint to W'

8. Complete your Venn Diagram by listing the elements of each subset in the appropriate circle.



Homework

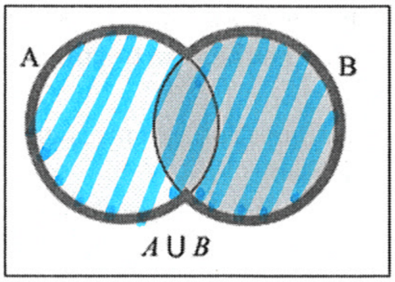
Lesson 2- Venn Diagrams & Cardinality (2 Circles)

Venn Diagrams: To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, uilding on a similar idea used by Leonhard Euler in the 18th century. These illustrations now called **Venn Diagrams**.

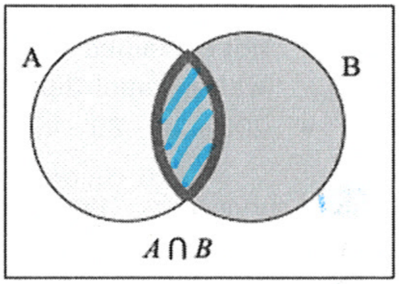
- A **Venn diagram** represents each set by a circle, usually drawn inside of a containing box representing the universal set. Overlapping areas indicate elements common to both sets.

EX: Create Venn diagrams to illustrate $A \cup B$, $A \cap B$, and $A' \cap B$

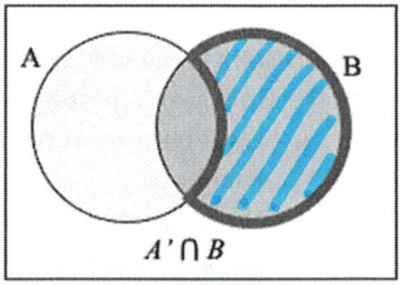
- $A \cup B$ contains all elements in *either* set.



- $A \cap B$ contains only those elements in both sets – in the overlap of the circles.



- A' will contain all elements *not* in the set A. $A' \cap B$ will contain the elements in set B that are not in set A.



Cardinality: Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

- The number of elements in a set is the cardinality of that set.
- The cardinality of the set A is often notated as $|A|$ or $n(A)$

EX: Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

- What is the cardinality of B? $A \cup B$, $A \cap B$?
    - The cardinality of B is 4, since there are 4 elements in the set.
    - The cardinality of $A \cup B$ is 7, since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements.
    - The cardinality of $A \cap B$ is 3, since $A \cap B = \{2, 4, 6\}$, which contains 3 elements.

EX: What is the cardinality of P = the set of English names for the months of the year?

- The cardinality of this set is 12, since there are 12 months in the year.

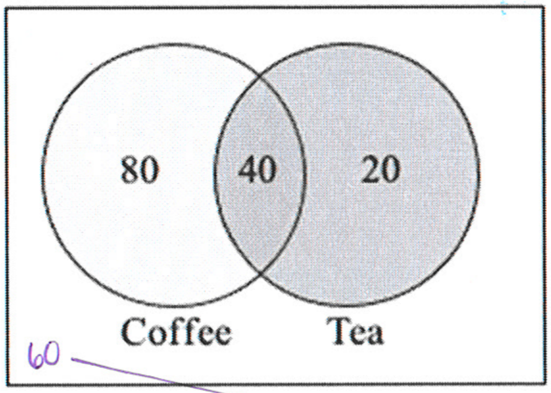
Sometimes we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying.

EX: A survey asks 200 people "What beverage do you drink in the morning", and offers choices:

Tea only
Coffee only
Both coffee and tea

↓ important!

Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea or coffee?



this is where the 60 belongs.

This question can most easily be answered by creating a Venn diagram.

- We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both: 60 people.
- We can also see that those who drink neither are those not contained in the any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.
- $\underbrace{200 - 20 - 80 - 40}_{\text{common practice}} = 60$ people who drink neither.

EX: A survey asks: Which online services have you used in the last month:

Twitter
Facebook
Have used both

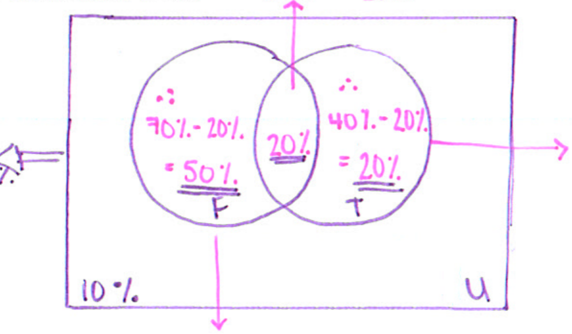
The results show 40% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. How many people have used neither Twitter or Facebook?

- Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook.
- Notice that while the cardinality of F is 70% and the cardinality of T is 40%, the cardinality of $F \cup T$ is not simply 70% + 40%, since that would count those who use both services twice.
- To find the cardinality of $F \cup T$, we can add the cardinality of F and the cardinality of T, then subtract those in intersection that we've counted twice. In symbols,

$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$
$$n(F \cup T) = 70\% + 40\% - 20\% = 90\%$$

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)^c$.

- Since the universal set contains 100% of people and the cardinality of $F \cup T$ = 90%, the cardinality of $(F \cup T)^c$ must be the other 10%.



The previous example illustrated two important properties

Cardinality properties

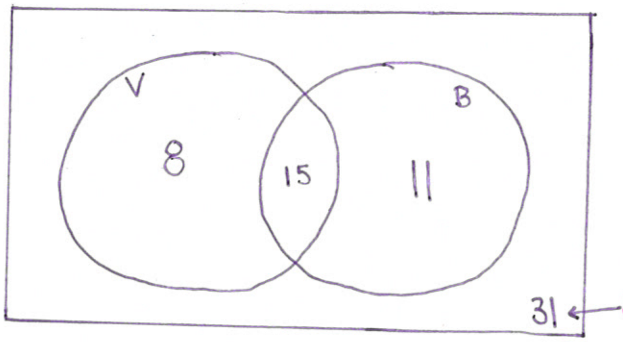
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A') = n(U) - n(A)$$

- Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

EX: In an Alberta school, there are 65 Grade 12 students. Of these students, 23 play volleyball and 26 play basketball. There are 31 students who do not play either sport. Complete a Venn Diagram to display this particular situation.



1. 23 + 26 + 31 = 80

* since only 65 surveyed + 80 answers, there has to be overlap *

2. 65 - 31 = 34 play n(V∪B)

3. Find n(V∩B)

$n(V \cap B) = n(V) + n(B) - n(V \cup B)$

$= 23 + 26 - 34 = 15$

31 ← can be placed w̄out any math.

⇓ place in the 'eye'

4. Solve for the rest of n(V)

23 - 15 = 8 ⇒ place in the V 'moon'

5. Solve for the rest of n(B)

26 - 15 = 11 ⇒ place in the B 'moon'

6. Add : 8 + 15 + 11 + 31 = 65 ✓

1. Are the sets disjoint?

   No! They overlap

2. How many students play both sports?

   15

3. How many play just volleyball?

   8

4. Just Basketball?

   11

5. How many play volleyball?

   8 + 15 = 23

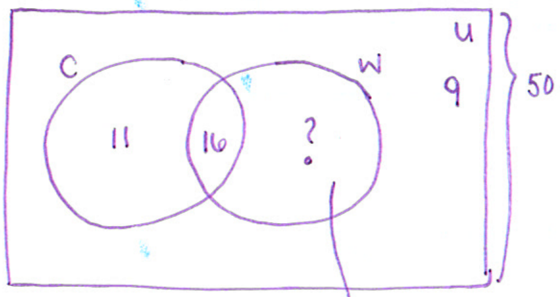
EX: Jamaal surveyed 50 gym members. Of these, 9 train fewer than three times a week, 11 just do cardio training 3 times per week, and 16 do both cardio and weights three times per week. Determine how many of these members just do weight training three times per week ~~using the formula.~~ Then compare using a Venn Diagram.

$n(U) = 50$

$n(W \cup C)' = 9$

$n(W' \cap C) = 11$

$n(W \cap C) = 16$



$11 + 16 + 9 = 36$

$50 - 36 = \underline{\underline{14}}$

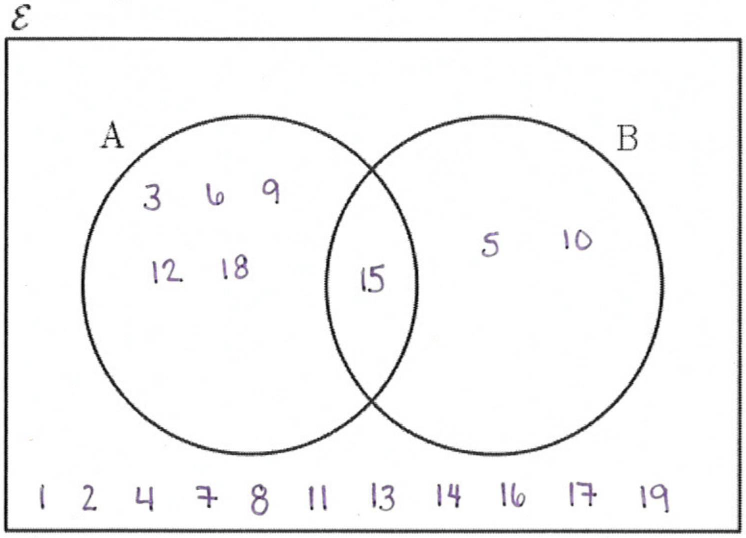
EX: Consider the following two sets of numbers:

A= {natural numbers less than 20 that are divisible by 3} → 3, 6, 9, 12, 15, 18

B= {natural numbers less than 20 that are divisible by 5} → 5, 10, 15

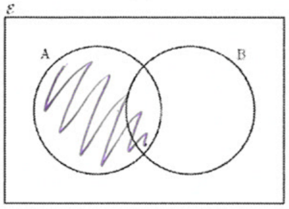
U= {natural numbers less than 20} → 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

Place them within this Venn Diagram:


ε
A
B
3 6 9
12 18
15
5 10
1 2 4 7 8 11 13 14 16 17 19

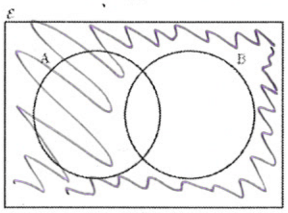
For the following sets, shade the region that is indicated and state the cardinality for that set.

A



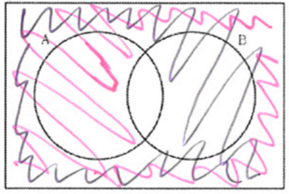
n(A) = 6

B'



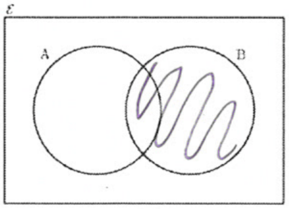
n(B') = 17

A'∪B' everything!



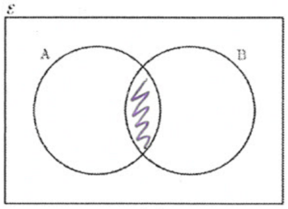
n(A'∪B') = 18

B



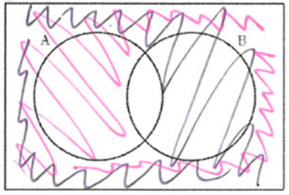
n(B) = 3

A∩B



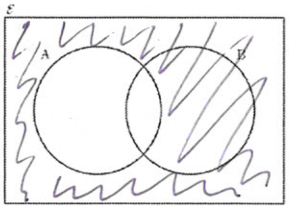
n(A∩B) = 1

A'∩B' overlap of colours!



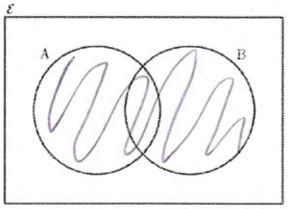
n(A'∩B') = 11

A'



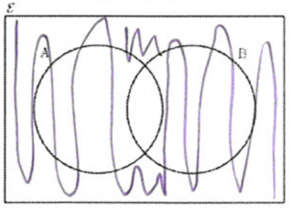
n(A') = 13

A∪B



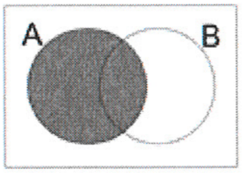
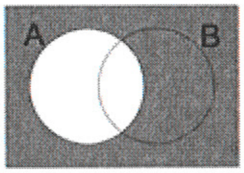
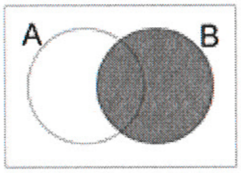
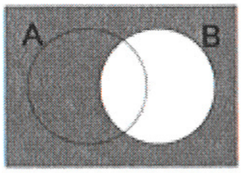
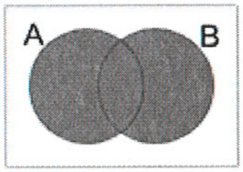
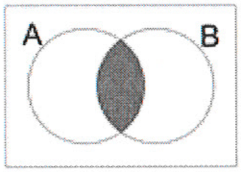
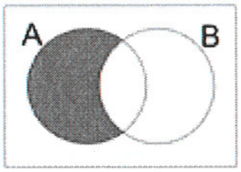
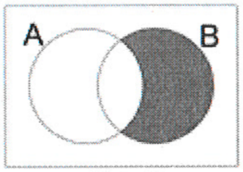
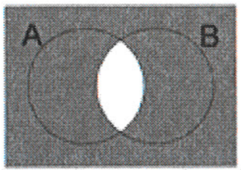
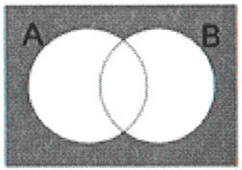
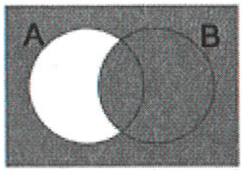
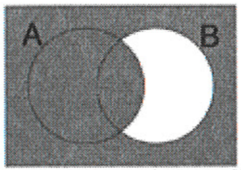
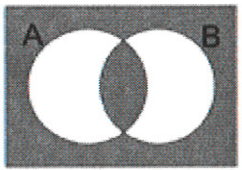
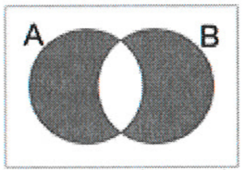
n(A∪B) = 8

(A∩B)'



n(A∩B)' = 18

2 Circle Venn Diagram Shading

 A	 A'	 B
 B'	 $A \cup B$	 $A \cap B$
 $A \cap B'$	 $A' \cap B$	 $(A \cap B)'$ or $A' \cup B'$
 $A' \cap B'$ or $(A \cup B)'$	 $A' \cup B$ or $(A \cap B')'$	 $A \cup B'$ or $(A' \cap B)'$
 $(A' \cap B') \cup (A \cap B)$	 $(A \cap B') \cup (A' \cap B)$	

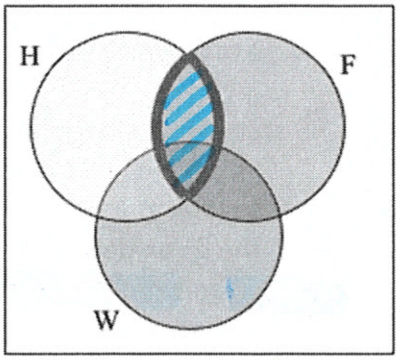
Homework

Lesson 3- Venn Diagrams & Cardinality (3 Circles)

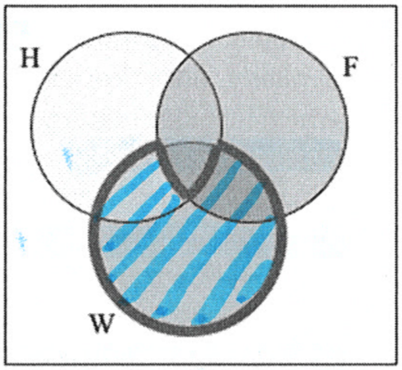
In this lesson, we are simply progressing into 3 circles instead of 2.

EX: Use a Venn diagram to illustrate $(H \cap F)' \cap W$

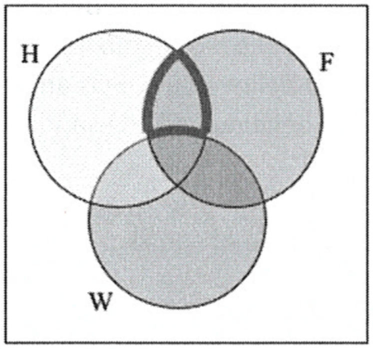
- We'll start by identifying everything in the set $H \cap F$



- Now, $(H \cap F)' \cap W$ will contain everything *not* in the set identified above that is also in set W.

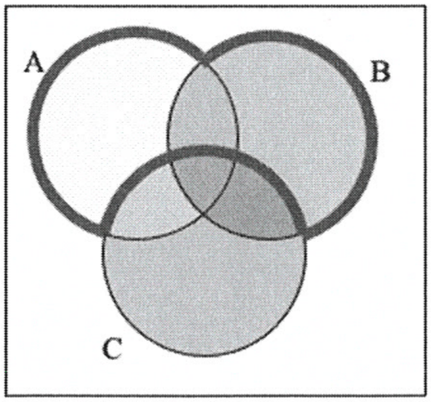


EX: Create an expression to represent the outlined part of the Venn diagram shown.



- The elements in the outlined set *are* in sets H and F, but are not in set W.
- So we could represent this set as $H \cap F \cap W'$

EX: Create an expression to represent the outlined portion of the Venn diagram shown



1. $A \cup B$, but not touching C

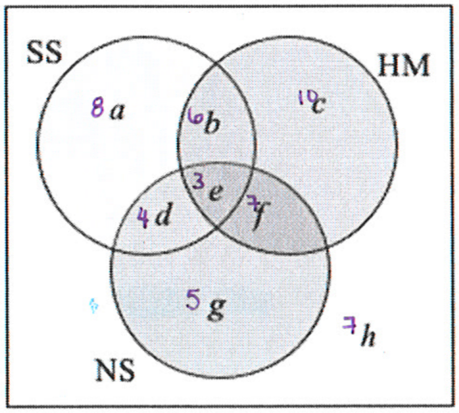
$\therefore$ $\underline{A \cup B \cap C'}$

EX: Fifty students were surveyed, and asked if they were taking a social science (SS), humanities (HM) or a natural science (NS) course the next quarter.

21 were taking a SS course
19 were taking a NS course
7 were taking SS and NS
start * 3 were taking all three
here!
Then work out!!

26 were taking a HM course
9 were taking SS and HM
10 were taking HM and NS
7 were taking none

How many students are only taking a SS course?



It might help to look at a Venn diagram.

- From the given data, we know that there are 3 students in region *e* and 7 students in region *h.*
- Since 7 students were taking a SS and NS course, we know that n(*d*) + n(*e*) = 7. Since we know there are 3 students in region 3, there must be 7 – 3 = 4 students in region *d.*
- Similarly, since there are 10 students taking HM and NS, which includes regions *e* and *f*, there must be 10 – 3 = 7 students in region *f.*
- Since 9 students were taking SS and HM, there must be 9 – 3 = 6 students in region *b.*
- Now, we know that 21 students were taking a SS course. This includes students from regions *a, b, d,* and *e*. Since we know the number of students in all but region *a*, we can determine that 21 – 6 – 4 – 3 = 8 students are in region *a.*
  - ○ 8 students are taking only a SS course.

* HM = 26-6-3-7 = 10(c)   * NS = 19-4-3-7 = 5(g)

EX: One hundred fifty people were surveyed and asked if they believed in UFOs, ghosts, and Bigfoot.

✓43 believed in UFOs
✓25 believed in Bigfoot
✓8 believed in ghosts and Bigfoot
✓2 believed in all three

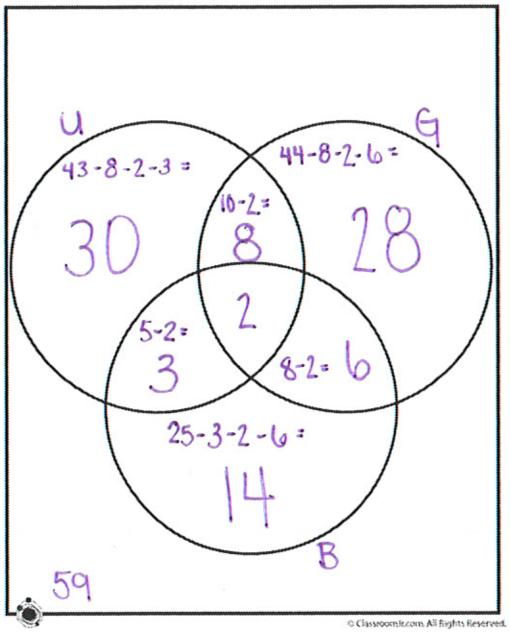
✓44 believed in ghosts
✓10 believed in UFOs and ghosts
✓5 believed in UFOs and Bigfoot

How many people surveyed believed in at least one of these things?

150 - 30 - 8 - 28 - 3 - 2 - 6 - 14 = 59 don't believe

∴

91 believe in at least 1.



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EX: Consider the following three sets of numbers:

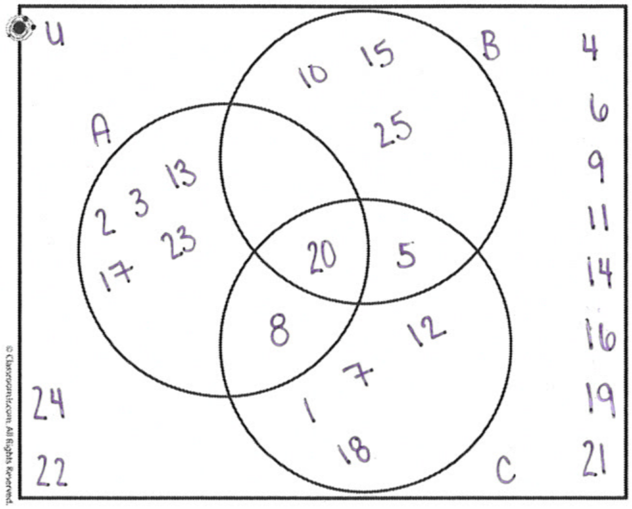
A= {2, 3, 8, 13, 17, 20, 23}

B= {5, 10, 15, 20, 25}

C= {1, 5, 7, 8, 12, 18, 20}

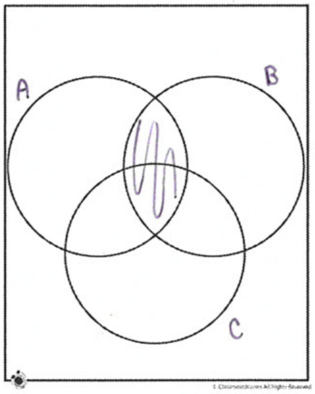
U= {Natural numbers 1 to 25}

Place them within this Venn Diagram:


U
A
B
C
2 3 13
17 23
10 15
25
20
5
8
12
7
1
18
24
22
4
6
9
11
14
16
19
21

For the following sets, shade the region that is indicated and state the cardinality for that set.

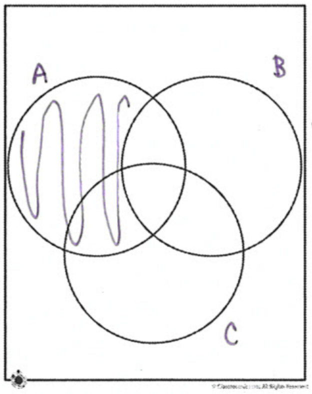
A and B



$n(A\cap B)$

= 1

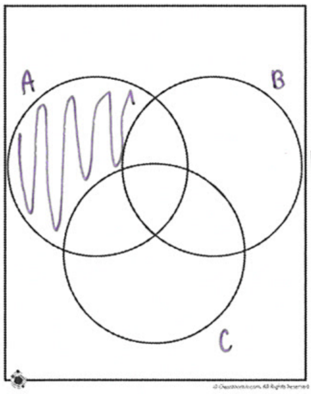
A and not B



$n(A\cap B')$

= 6

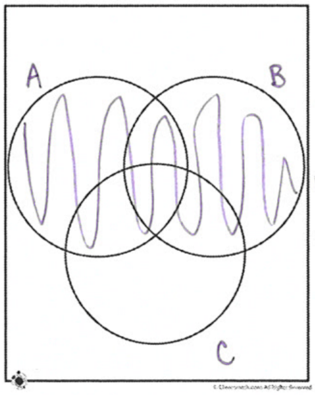
A and not B and not C



$n(A\cap B'\cap C'$

= 5

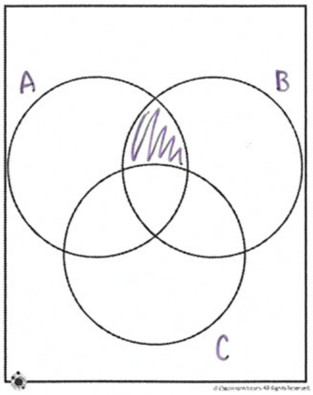
A or B



$n(A\cup B)$

= 11

A and B and not C

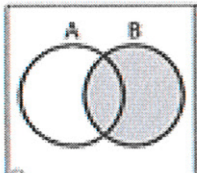


$n(A\cap B\cap C'$

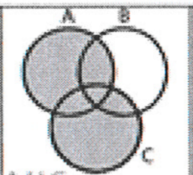
= 0

Unions, Intersections, and Complements
(Name the shaded regions)

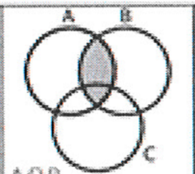
Name Answer Key



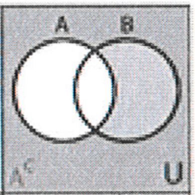
B



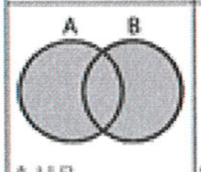
$A \cup C$



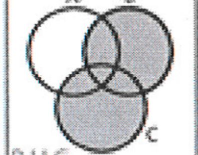
$A \cap B$



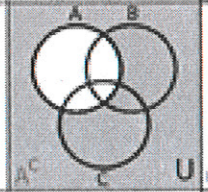
A^c



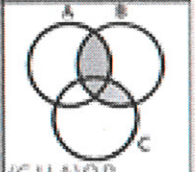
$A \cup B$



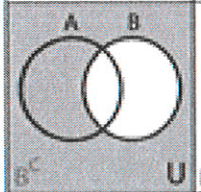
$B \cup C$



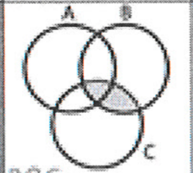
A^c



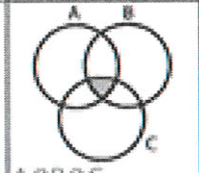
$(C \cup A) \cap B$



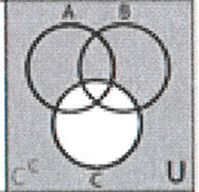
B^c



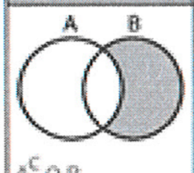
$B \cap C$



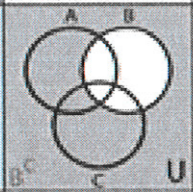
$A \cap B \cap C$



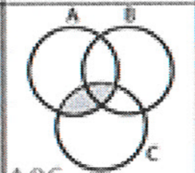
C^c



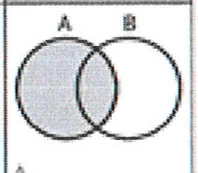
$A^c \cap B$



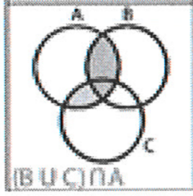
B^c



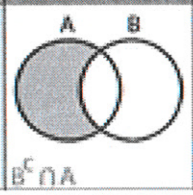
$A \cap C$



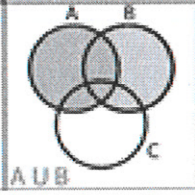
A



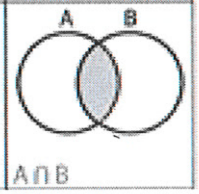
$(B \cup C) \cap A$



$B^c \cap A$



$A \cup B$



$A \cap B$

Homework

Unit 2- Probability

Lesson 4- Probability Fundamentals

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability.

- One would be **experimental** in nature, where we *repeatedly conduct an experiment.* Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.
- Another view would be **subjective** in nature, in other words an *educated guess*. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.
- We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the probability of that event is defined as

$$\frac{m}{n}$$

Event: An **event** is any particular outcome or group of outcomes.

Outcome: The result of an experiment is called an **outcome**.

Sample space: The **sample space** is the set of all possible simple events.

EX: If we roll a standard 6-sided die, describe the sample space and some simple events.

- The sample space is the set of all possible simple events: {1,2,3,4,5,6}
- Some examples of simple events:
  - We roll a 1
  - We roll a 5
- Some compound events:
  - We roll a number bigger than 4
  - We roll an even number

Basic Probability: Given that all outcome are equally likely, we can compute the probability of an event using this formula:

$$Probability\ of\ an\ Event = \frac{Number\ of\ ways\ an\ event\ CAN\ occur}{Total\ number\ of\ possible\ outcomes}$$

OR

IMPORTANT !! $\left\{ P(E) = \frac{Favourable}{Total} \right.$ ← This is NOT on your formula sheet.

EX: If we roll a 6-sided die, calculate

a) P(rolling a 1)

b) P(rolling a number bigger than 4)

Recall that the sample space is {1,2,3,4,5,6}

- A- There is one outcome corresponding to "rolling a 1", so the probability is

$$\frac{\mathbf{1}}{\mathbf{6}}$$

- B- There are two outcomes bigger than a 4, so the probability is

$$\frac{2}{6} = \frac{\mathbf{1}}{\mathbf{3}}$$

Note: Probabilities are essentially fractions, and can be ***reduced to lower terms*** like fractions.

*IMPORTANT - SHOW CALC TRICK! *

EX: Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

- There are 20 possible cherries that could be picked, so the number of possible outcomes is 20.
- Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is

$$\frac{14}{20} = \frac{\mathbf{7}}{\mathbf{10}}$$

Note: There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.)

- Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, ***we rely heavily on the assumption of equal probability for all outcomes***.

EX: At some random moment, you look at your clock and note the minutes reading.

a. What is probability the minutes reading is 15?

$$\frac{\text{how many 15's?}}{\text{how many options}} = \underline{\underline{\frac{1}{60}}}$$

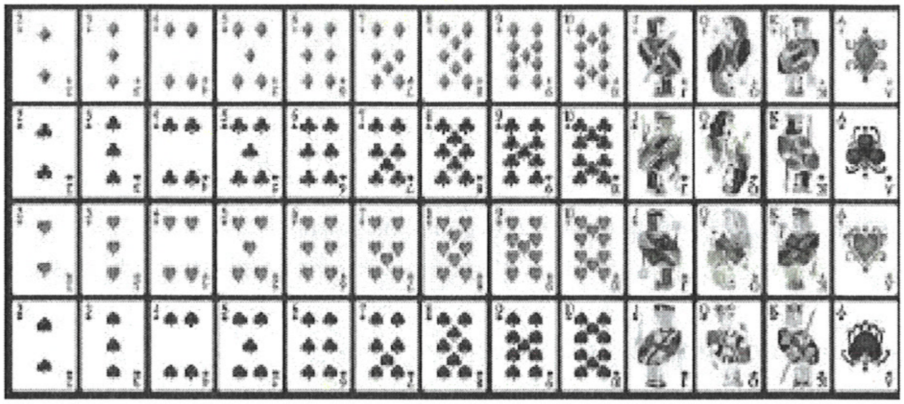
b. What is the probability the minutes reading is 15 or less?

this includes 15 ∴ $\frac{15}{60} = \underline{\underline{\frac{1}{4}}}$

Cards: A standard deck of 52 playing cards consists of the following:

IMPORTANT

- Four suits (hearts, spades, diamonds and clubs).
- Spades and clubs are black while hearts and diamonds are red.
- Each suit contains 13 cards, each of a different rank:
    - An Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.



EX: Compute the probability of randomly drawing one card from a deck and getting an Ace.

- There are 52 cards in the deck and 4 Aces so

$$P(Ace) = \frac{4}{52} = \frac{1}{13} \approx \mathbf{0.0769}$$

- We can also think of ***probabilities as percents***: * IMPORTANT *
    - There is a 7.69% chance that a randomly selected card will be an Ace.

IMPORTANT:

- Notice that the smallest possible probability is 0 (0%) – if there are no outcomes that correspond with the event. The largest possible probability is 1 (100%) – if all possible outcomes correspond with the event.

*IMPORTANT

- The probability of any event must be $0 \leq P(E) \leq 1$
    - In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Impossible Event: An impossible event has a probability of 0.

Certain Event: A certain event has a probability of 1.

Complementary Events: Now let us examine the probability that an **event does not happen**. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(six) = \frac{1}{6}$.

- Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is

$$P(not\ a\ six) = \frac{5}{6}$$

- Notice that

$$P(six) + P(not\ a\ six) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

- The **complement** of an event is the event "*E* doesn't happen"
  - The notation $\bar{E}$ or E^c or E' is used for the complement of event *E*. This should look familiar from Unit 1.
- We can compute the probability of the complement using

$$P(\bar{E}) = 1 - P(E)$$

- Notice also that

$$P(E) = 1 - P(\bar{E})$$

EX: If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

- There are 13 hearts in the deck, so

$$P(heart) = \frac{13}{52} = \frac{1}{4}$$

- The probability of *not* drawing a heart is the complement:

$$P(not\ heart) = 1 - P(heart) = 1 - \frac{1}{4} = \frac{\mathbf{3}}{\mathbf{4}}$$

You don't have to do it this way! You can just calculate P(ϕ heart) *

EX: Visualize tossing a 6-sided die.

a) What is the theoretical probability of getting a 5?

$$\frac{1}{6}$$

b) What is the theoretical probability of getting an even number?

$$\frac{3}{6} = \frac{1}{2}$$

c) What is the theoretical probability of getting a number $\underbrace{\text{greater than}}_{\text{ϕ include 2}}$ 2?

$$\frac{4}{6} = \frac{2}{3}$$

Homework

Lesson 5- Odds & Expected Events

Calculating Odds: Mathematically probability and odds mean two different things, and are expressed differently.

- As you now know, a probability is a fraction representing the number of ways a particular event can occur divided by the total number of possible outcomes.
- The odds of an event are given by the ratio of the number of times the event occurs to the number of times the event does not occur.

Odds FOR (in favor of) an Event:

$$Odds = number\ of\ ways\ event\ CAN\ occur : number\ of\ ways\ even\ CANNOT\ occur$$

OR

IMPORTANT * $$\left\{ Odds = Favorable : Unfavorable \right.$$ ← Not on your formula sheet

Note: To avoid confusion with probability, odds are usually left as a ratio such as $1:5$, which would be read as "**one to five**". When probability is read as a ratio, it's usually written as a fraction like $\frac{1}{5}$, which would usually be read as "**one in five**."

EX: Find the odds for the event of tossing a coin and getting heads.

- The key to finding odds is looking at how many outcomes result in the event and how many do not.
- The sample space consists of 2 outcomes: heads or tails.
    - The event we are interested is the event that the coin lands on heads.
        - There is 1 way the event can occur (the coin lands on heads) and there is 1 way the event can NOT occur (the coin lands on tails).

$$\boldsymbol{The\ odds\ for\ getting\ heads\ are\ 1 : 1}$$

Note: We call those odds ***one to one, or even***. We are just as likely to get a head as to not get a head.

EX: Find the odds for rolling a die and getting a 3.

- Odds for getting a 3 = number of ways to get a 3 : number of ways to not get a 3
- Odds for getting a 3 = **1 : 5**

IMPORTANT: Look carefully at the example above. This illustrates the need to avoid confusion between odds and probability.

- We know that the probability of getting a 3 is $P(3) = 1/6$ or "one in six"
- The odds describes the same event with the ratio $1 : 5$ or "one to five"

EX: A board game has a spinner that is divided into 8 different colored sections. The sections are red, orange, yellow, green, blue, purple, black, and white. Find the odds of the spinner landing on a primary color (red, yellow, or blue).

$\underbrace{\underline{\underline{3:5}}}$ * check reduction in calc*

$3+5 = 8 \Rightarrow$ sample space

EX: You are told that the probability of a tornado hitting your hometown during the month of May is 0.15. Find the odds for a tornado hitting your hometown during May.

$$Probability\ of\ tornado = 0.15\ or\ \frac{15}{100}$$

- This means *favorable = 15* and *total = 100*
- *Unfavorable = 100-15 =85* * IMPORTANT

- Odds = Favorable : Unfavorable
- Odds = 15 : 85
- Now ***reduce*** by dividing both sides by 5 * OR CALC TRICK *

$$\boldsymbol{Odds = 3 : 17}$$

EX: Find the odds for tossing 4 coins and getting exactly 3 tails.

- In this example, we first need to determine what all of the possible outcomes are when you toss 4 coins. Here is a list of all possible outcomes when tossing 4 coins:

HHHH
THTH
TTTT
HTHT
HHHT
TTHT
TTTH
HHTH
HHTT
THHT
TTHH
HTTH
HTTT
HTHH
THHH
THTT

- There are 4 outcomes where exactly 3 tails came up and 12 outcomes when they did not.

$$\boldsymbol{So\ the\ odds\ for\ getting\ 3\ tails\ are\ 4 : 12 = 1 : 3\ (one\ to\ three).}$$

Calculating Odds Against an Event: Most gambling applications involve calculating and reporting the odds against an event. The odds against an event are given by a *ratio,* just like the odds for an event.

- Since we now want the odds against an event occurring, we create a ratio of the number of times the event *does not* occur to the number of times the event *does* occur.

Odds Against an Event:

$$Odds = number\ of\ ways\ event\ CANNOT\ occur : number\ of\ ways\ event\ CAN\ occur$$

OR

$$Odds = Unfavorable : Favorable$$ ← Notice all we did was swap them!

EX: Find the odds against tossing 4 coins and getting exactly 3 tails.

- In the previous example we calculated the odds for this event. Now we want to calculate the odds against it.
- Recall, the sample space contains 4 outcomes where exactly 3 tails came up and 12 outcomes when they did not.
- Since we are looking for the odds against tossing 4 coins and getting exactly 3 tails, we need to take the ratio of the number of ways we do not get exactly 3 tails to the number of ways we do get exactly 3 tails.

$$\boldsymbol{Odds\ against\ getting\ exactly\ 3\ tails = 12 : 4 = 3 : 1\ (three\ to\ one).}$$

EX: Experts calculate the probability of a particular horse winning the Kentucky Derby to be 0.2. Calculate the odds against the horse winning the race.

$$Probability\ of\ winning = 0.2\ or\ \frac{2}{10}\ or\ \frac{1}{5}$$

MATH, FRAC, ENTER

- This means *favorable = 1* and *total = 5*
- *Unfavorable = 5-1 =4*

- Odds(against) = Unfavorable : Favorable

$$\boldsymbol{Odds = 4 : 1}$$

EX: Find the odds for drawing a club from a standard 52-card deck.

$13 : 39 \longrightarrow \underline{\underline{1:3}}$

EX: Find the odds against drawing a club from a standard 52-card deck.

$\underline{\underline{3:1}}$

EX: Suppose that the probability of an event happening is $\frac{2}{5}$.

a) What are the odds in favour of the event happening?

$\underline{\underline{2:3}}$

b) What are the odds against the event happening?

$\underline{\underline{3:2}}$

EX: Suppose the odds in favour of an event are 5:3. Is the probability that the event will happen greater or less than 50%? Support your answer by determining the probability of the event occurring.

$\frac{5}{total} = \frac{5}{5+3} = \frac{5}{8} = \underline{\underline{62.5\%}}$, yes, the probability is greater then 50%.

EX: The grade 12 students are holding a carnival as a fundraiser for graduation. They are still trying to decide on one more game, and are considering game A and game B. The odds against winning game A are 11:3 and the odds against winning game B are 17:6. As the students are trying to make as much money as possible, they would prefer not to give out as much in prizes. Which game should the grade 12's choose?

	ODDS TO WIN	PROBABILITY	PERCENT
A	3 : 11	$\frac{3}{14}$	21%
B	6 : 17	$\frac{6}{23}$	26% GAME B is better

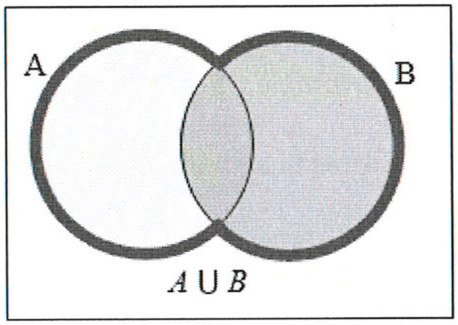
Homework

Lesson 6- And/ Or Probabilities

Lesson 6A- Mutually Exclusive Events and Event $A \cup B$

P(A or B): The probability of either *A* or *B* occurring (or both) is

$$P(A\ or\ B) = P(A) + P(B) \underbrace{-\ P(A\ and\ B)}_{\text{because you count it twice!}}$$



- This property is on your formula sheet but is written as:
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Remember that
  - "$A \cup B$" means A Union B
  - "$A \cap B$" means A Intersection B

EX: Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

- Here, there are still 12 possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}
- By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\frac{7}{12}$.
  - How could we have found this from ***the individual probabilities***?
- As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is ***not the correct probability***.
  - Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6;
  - The probability of both a head *and* rolling a 6 is $\frac{1}{12}$.
  - If we subtract out this double count, we have the correct probability:

$$\frac{8}{12} - \frac{1}{12} = \mathbf{\frac{7}{12}}$$

EX: Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

- Half the cards are red, so $P(Red) = \frac{26}{52}$
- There are four kings, so $P(King) = \frac{4}{52}$
- There are two red kings, so $P(Red\ and\ King) = \frac{2}{52}$
- We can then calculate

$$P(Red\ or\ King) = P(Red) + P(King) - P(Red\ and\ King) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \mathbf{\frac{28}{52}}$$

EX: In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

$P(WSock) = \frac{6}{10} = \frac{3}{5}$

$P(WTee) = \frac{3}{7}$

$P(WS \cap WT) = \frac{9}{35}$ *Give to them!!

$P(WS \cup WT) = P(WS) + P(WT) - P(WS \cap WT)$

$= \frac{3}{5} + \frac{3}{7} - \frac{9}{35} = \underline{\underline{\frac{27}{35}}}$

↳ learn in 6B → INPED EVENTS! $P(WS \cap WT) = P(WS) \cdot P(WT)$

Mutually Exclusive Events: The probability of one event does not affect the other. They are ***DISJOINT*** events.

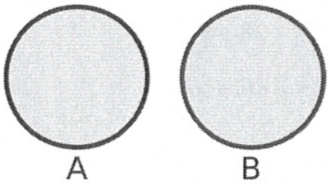
- Drawing an ace or flipping a head.
- There is **no overlap** of the probabilities, therefore, we can ignore the last part of the equation (they can be visualized as disjoint sets in a Venn Diagram).

$$P(A\ or\ B) = P(A) + P(B)\ \cancel{- P(A\ and\ B)}$$

↳ = 0!!

And get

$$P(A\ or\ B) = P(A) + P(B)$$



- This property is on your formula sheet but is written as

$$P(A \cup B) = P(A) + P(B)$$

- **IMPORTANT:** When you have Mutually Exclusive events, the sum of the two events has to be **less than or equal to one**. If the sum of the events is over one, the events are NOT mutually exclusive.

EX: Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

- There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(King\ or\ Queen) = \frac{8}{52}$$

- Note that in this case, there are no cards that are both a Queen and a King, so

$$P(King\ and\ Queen) = 0$$

- Using our probability rule, we could have said:

$$P(King\ or\ Queen) = P(King) + P(Queen) - P(King\ and\ Queen) = \frac{4}{52} + \frac{4}{52} - 0 = \mathbf{\frac{8}{52}}$$

EX: Blocks of different shapes and colours are placed into a bag:

- There are 5 Cubes: one-red, one-green, one-yellow, one-black, one-white
- There are 3 Spheres: one-red, two-blue
- There are 2 Pyramids: one-blue, one-green

} 10 total

Determine the probability that a block selected at random

a) Will be a $\underbrace{\text{sphere or a cube}}_{\text{ME!}}$:

$\therefore\ P(S \cup C) = P(S) + P(C) = \frac{3}{10} + \frac{5}{10} = \frac{8}{10} = \underline{\underline{\frac{4}{5}}}$

b) Will not be green:

$1 - P(green) = 1 - \frac{1}{5} = \underline{\underline{\frac{4}{5}}}$

c) Will be $\underbrace{\text{blue or a sphere}}_{\phi\text{ME}}$

$\therefore\ P(B \cup S) = P(B) + P(S) - P(B \cap S)$

$= \frac{3}{10} + \frac{3}{10} - \frac{2}{10} = \frac{4}{10} = \underline{\underline{\frac{2}{5}}}$

Summary for Finding $P(A \cup B)$:

- **Mutually Exclusive Events**: Disjoint events (the Venn circles do NOT overlap)
  - $P(A \cup B) = P(A) + P(B)$
- **Non-Mutually Exclusive Events:** Events that overlap
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

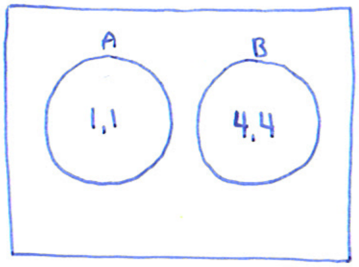
} *IMPORTANT

EX: Zach is playing a board game. He must roll two four-sided dice, numbered 1 to 4. He can move if he rolls a sum of 2 or a sum of 8.

a) Use A and B to represent the two events that will allow Zach to move. Then draw a Venn diagram to illustrate A and B.

Sum of 2: $A = \{1,1\}$

Sum of 8: $B = \{4,4\}$



b) Are A and B mutually exclusive or not mutually exclusive?

Yes, mutually exclusive.

c) Determine the probability that Zach will roll a sum of 2 or a sum of 8.

total options $= 4 \cdot 4 = 16$

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \underline{\underline{\frac{1}{8}}}$$

d) Determine the probability that Zach will roll doubles or a sum of 6.

* NOT ME anymore!

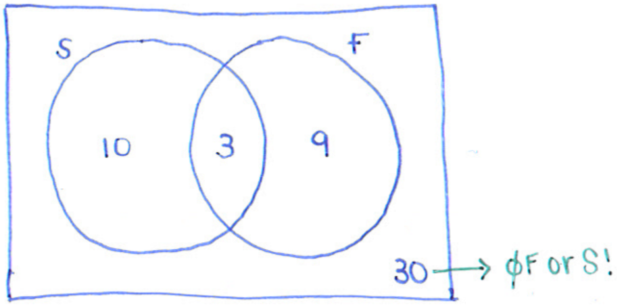
$\{2,4\}\{4,2\}\{3,3\}$

$P(D) = \frac{4}{16} = \frac{1}{4}$, $P(6) = \frac{3}{16}$, $P(D \cap 6) = \frac{1}{16}$ $\therefore P(D \cup 6) = P(D) + P(6) - P(D \cap 6)$

$$= \frac{1}{4} + \frac{3}{16} - \frac{1}{16} = \frac{6}{16} = \underline{\underline{\frac{3}{8}}}$$

EX: Pearl is about to draw a card at random from a standard deck of 52 playing cards. If she draws a face card or a spade, she will win a point.

a) Draw a Venn diagram to represent the two events.



b) Are the events mutually exclusive?

No, because there are Face cards that are spades.

c) Determine the probability of drawing a face card or a spade.

$P(F) = \frac{12}{52} = \frac{3}{13}$

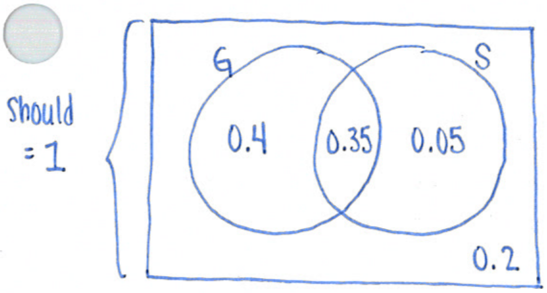
$P(S) = \frac{13}{52} = \frac{1}{4}$

$P(S \cap F) = \frac{3}{52}$

$$P(S \cup F) = P(S) + P(F) - P(S \cap F)$$
$$= \frac{3}{13} + \frac{1}{4} - \frac{3}{52} = \underline{\underline{\frac{11}{26}}}$$

EX: The probability that Maria will go to the gym on Saturday is 0.75. The probability that she will go shopping on Saturday is 0.4. The probability that she will do neither is 0.2.

a) Draw a Venn diagram to represent the two events.



$$\left.\begin{array}{l} P(G \cup S) = 0.8 \\ P(G) = 0.75 \\ P(S) = 0.4 \end{array}\right\}$$

$$P(G \cap S) = P(G) + P(S) - P(G \cup S)$$
$$= 0.75 + 0.4 - 0.8$$
$$= 0.35$$

b) Are the two events mutually exclusive?

No!

c) Determine the probability that Maria will do at least one of these activities on Saturday.

$P(G \cup S) = \underline{\underline{0.8}}!$

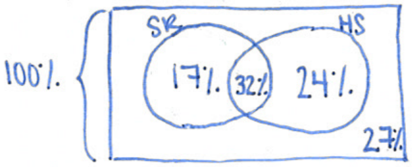
Take $1 - 0.2 = 0.8$

EX: A database reports that 56% of the cars on a dealership lot have heated leather seats, 49% have a sunroof, and 27% have neither.

a) Are the events mutually exclusive? Prove it.

$P(HS \cup SR) = P(SR) + P(HS) = 56\% + 49\% = 105\%$ ∴ not ME b/c sum is over 100%.

b) If non – mutually exclusive, what is the probability that a car at the dealership has heated leather seats and a sunroof?



$P(HS \cup SR) = 100\% - 27\% = 73\%$

$$\begin{array}{r} P(HS) + P(SR) = 105\% \\ \underline{-73\%} \\ 32\% \end{array}$$

over ∴ $\underline{\underline{P(HS \cap SR) = 32\%}}$

c) What is the probability of a car having a sunroof but no heated leather seats?

$P(SR \underset{\text{not}}{\underset{\downarrow}{\setminus}} HS) = \underline{\underline{17\%}}$

Homework- 6A

Lesson 6B- Independent vs. Dependent Events

Probability of Two Independent Events: Events A and B are independent events if the probability of Event B occurring is the same whether or not Event A occurs.

- The events do NOT impact each other!

EX: Are these events independent?

a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

- The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

- These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) You draw a card from a deck, then draw a second card without replacing the first.

- The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

P(A and B) for Independent Events: When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

- If events *A* and *B* are independent, then the probability of both *A* and *B* occurring is

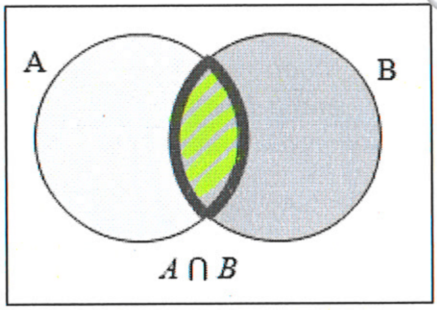
$$P(A\ and\ B) = P(A) \cdot P(B)$$

- where *P*(*A* and *B*) is the probability of events *A* and *B* both occurring, *P*(*A*) is the probability of event *A* occurring, and *P*(*B*) is the probability of event *B* occurring

- This property is on your formula sheet but is written as

$$P(A \cap B) = P(A) \cdot P(B)$$

  - This can be thought of as the ***intersection*** of P(A) and P(B)



EX: In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

- The probability of choosing a white pair of socks is $\frac{6}{10}$
- The probability of choosing a white tee shirt is $\frac{3}{7}$
- The probability of both being white is

$$\underbrace{\frac{6}{10} \cdot \frac{3}{7}}_{\text{this means the draws are indep.}} = \frac{18}{70} = \mathbf{\frac{9}{35}}$$

EX: A card is pulled a deck of cards and noted. **The card is then replaced**, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

$$P(A \cap A) = P(A) \cdot P(A)$$

$$= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{13} \cdot \frac{1}{13} = \underline{\underline{\frac{1}{169}}}$$

Conditional Probability: Often it is required to compute the probability of an event ***given that another event has occurred.***

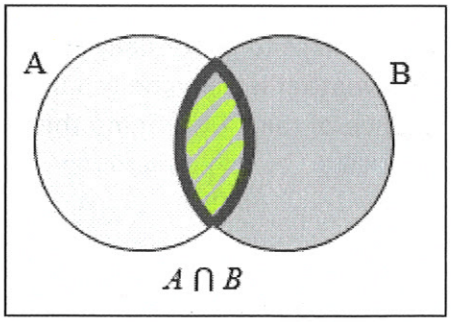
- The probability the event *B* occurs, given that event *A* has happened, is represented as
  $$P(B \mid A)$$
  - This is read as "the probability of *B* given *A*"

EX: What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

- It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.
- This would be ***incorrect***, however, because the ***two events are not independent***.
  - If the first card drawn is an ace, then the probability that the second card is also an ace would be lower because there would only be three aces left in the deck.
- Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the conditional probability of drawing an ace.
  - In this case the "condition" is that the first card is an ace. Symbolically, we write this as:
    $$P(ace\ on\ second\ draw \mid an\ ace\ on\ the\ first\ draw).$$
  - The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw."
- What is this probability?
  - After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left.
  - This means that the conditional probability of drawing an ace after one ace has already been drawn is: $\frac{3}{51} = \frac{1}{17}$
- Thus, the probability of both cards being aces is
  $$\mathbf{\frac{4}{52} \cdot \frac{1}{17} = \frac{1}{221}}$$

Conditional Probability Formula (Dependent Events): If Events *A* and *B* are not independent, then
$$P(A\ and\ B) = P(A) \cdot P(B \mid A)$$

- This property is on your formula sheet but is written as
  $$P(A \cap B) = P(A) \cdot P(B \mid A)$$
- This can still be thought of as the intersection of A and B, ***but B will now be different than it would have been because of event A***.



EX: If you pull 2 cards out of a deck, what is the probability that both are spades?

- The probability that the first card is a spade is $\frac{13}{52}$
- The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.
- The probability that both cards are spades is
  $$\frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} \approx \mathbf{0.0588}$$

EX: In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

$P(WS_1) = \frac{6}{10} = \frac{3}{5}$

$P(WS_2|WS_1) = \frac{5}{9}$

$P(WS_1 \cap WS_2) = P(WS_1) \cdot P(WS_2|WS_1)$

$= \frac{3}{5} \cdot \frac{5}{9} = \frac{15}{45} = \underline{\underline{\frac{1}{3}}}$

Summary for Finding $P(A \cap B)$:

- **Independent Events**: One event does NOT impact the next event from happening
  - $P(A \cap B) = P(A) \cdot P(B)$
- **Dependent Events:** One event depicts the probability of the second event from happening
  - $P(A \cap B) = P(A) \cdot P(B \mid A)$

*IMPORTANT

IMPORTANT:

- **Mutually Exclusive Vs. Independent:**
  - **Mutually Exclusive** involves whether or not two events can occur at the same time.
  - **Independent** involves whether or not the occurrence of one event has an effect on the probability of the other event occurring.

EX: Kyler has four loonies, three toonies, and five quarters in his pocket. He needs two loonies for a parking meter. He reaches into his pocket and pull out two coins at random. Determine the probability that both coins are loonies.

↓ dependent

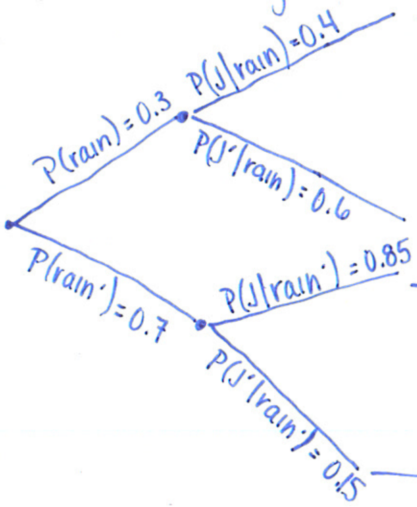
12 coins!

$P(L_1 \cap L_2) = P(L_1) \cdot P(L_2|L_1)$

$= \frac{4}{12} \cdot \frac{3}{11} = \underline{\underline{\frac{1}{11}}}$

EX: Ian likes to go for daily jogs with his dog, Oliver. If the weather is nice, he is 85% likely to jog for 8 km. If the weather is rainy, he is only 40% likely to jog 8 km. The weather forecast for tomorrow indicates a 30% chance of rain. Determine the probability that Ian will jog for 8 km.

* INDEP ∴ tree diagrams help!



$\rightarrow P(rain \cap J) = 0.3 \cdot 0.4 = 0.12$

$\rightarrow P(rain \cap J') = 0.3 \cdot 0.6 = 0.18$

$\rightarrow P(rain' \cap J) = 0.7 \cdot 0.85 = 0.595$

$\rightarrow P(rain' \cap J') = 0.7 \cdot 0.15 = 0.105$

1 * self check!

$P(Jog) = P(rain \cap Jog) + P(rain' \cap Jog)$

$= 0.12 + 0.595$

$= \underline{\underline{0.715}}$

EX: Cell phone users in Mapletown were surveyed about their phone plan with these results:

- 70% of all users have call display $P(CD) = 0.7$
- 40% of all users have a data plan $P(DP) = 0.4$
- 75% of users with a data plan also have call display. $P(CD|DP) = 0.75$ ∴ dependent.

A cell phone user in Mapletown, who is selected at random, has call display. Determine the probability that this person also has a data plan. $\Rightarrow P(DP|CD) \rightarrow$ NOT THE SAME AS $P(CD|DP)$

1. Find $P(CD \cap DP) = P(DP) \cdot P(CD|DP)$

$= 0.4 \cdot 0.75 = 0.3$

2. Find $P(DP|CD)$

$\hookrightarrow P(CD \cap DP) = P(CD) \cdot \underbrace{P(DP|CD)}_{\text{isolate}}$

$\therefore P(DP|CD) = \dfrac{P(CD \cap DP)}{P(CD)} = \dfrac{0.3}{0.7} = \underline{\underline{0.429}}$

EX: Celeste goes to the gym five days a week. Each day, she does a cardio workout using either a treadmill, an elliptical walker or a stationary bike. She follows this with a strength workout using either free weights or the weight machines. Celeste randomly chooses which cardio workout and which strength workout to do each day.

a) Are choosing a cardio workout and choosing a strength workout dependent or independent events? Explain.

$\downarrow$

independent b/c choosing cardio has no impact on choosing weights.

b) Determine the probability that Celeste will use a stationary bike and free weights the next day she goes to the gym.

$P(FW \cap SB) = P(FW) \cdot P(SB)$

$= \underbrace{\frac{1}{2}}_{\text{2 choices for weights}} \times \underbrace{\frac{1}{3}}_{\text{3 choices for cardio}} = \underline{\underline{\frac{1}{6}}}$

EX: Suppose the $P(A) = 0.25, P(B) = 0.52$ and $P(A \cup B) = 0.64$. Are A and B mutually exclusive? Or Independent events? Explain.

$\downarrow$

$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\text{isolate}}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.25 + 0.52 - 0.64 = 0.13 \leftarrow$ match!! ∴ indep.

verify indep: $P(A \cap B) = P(A) \cdot P(B)$

$= 0.25 \cdot 0.52$

$= 0.13$

$\downarrow$

$P(A \cup B) = P(A) + P(B)$

$= 0.25 + 0.52$

$= 0.77$ ∴ doesn't match question ∴ NOT ME!

EX: For each situation, classify the events as either independent or dependent. Justify your classification.

a) A four-colour spinner is spun, and a die is rolled. The first event is spinning red, and the second event is rolling a 2.

INDEP: rolls don't affect spins + vice versa.

b) A red die and a green die are rolled. The first event is rolling a 1 on the red die, and the second event is rolling a 5 on the green die.

INDEP: red die doesn't affect green die.

c) Two cards are drawn, without being replaced, from a standard deck of 52 playing cards. The first event is drawing a king, and the second event is drawing an ace.

DEPEND: "without replacement"

d) There are 30 cards, numbered 1 to 30, in a box. Two cards are drawn, one at a time, with replacement. The first event is drawing a prime number, and the second event is drawing a number that is a multiple of 5.

INDEP: "with replacement"

EX: Valeria draws a card from a well-shuffled standard deck of *52* playing cards. Then she draws another card from the deck without replacing the first card.

a) Are these two events dependent or independent?

DEPENDENT! "without replacement"

b) Determine the probability that both cards are diamonds.

$$P(D_1 \cap D_2) = P(D_1) \cdot P(D_2 | D_1)$$

$$= \frac{1}{4} \cdot \frac{12}{51} = \underline{\underline{\frac{1}{17}}}$$

EX: Valeria draws a card from a well-shuffled standard deck of *52* playing cards. Then she puts the card back in the deck, shuffles again, and draws another card from the deck.

a) Are these two events dependent or independent?

INDEPENDENT! "puts card back"

b) Determine the probability that both cards are diamonds.

$$P(D_1 \cap D_2) = P(D_1) \cdot P(D_2)$$

$$= \frac{1}{4} \cdot \frac{1}{4} = \underline{\underline{\frac{1}{16}}}$$

EX: Nathan asks Riel to choose a number between 1 and 40 and then say one fact about the number. Riel says that the number he chose is a multiple of 4. Determine the probability that the number is also a multiple of 6.

Let A be multiples of 4 = $\{4, 8, \underline{12}, 16, 20, \underline{24}, 28, 32, \underline{36}, 40\}$

Let B be multiples of 6 = $\{6, \underline{12}, 18, \underline{24}, 30, \underline{36}\}$

$P(A \cap B) = \frac{3}{40}$

$P(A) = \frac{10}{40} = \frac{1}{4}$

want $P(B|A)$ ∴ $P(A \cap B) = P(A) \cdot \underbrace{P(B|A)}_{\text{isolate}}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3/40}{1/4} = \underline{\underline{\frac{3}{10}}}$$

EX: According to a survey, 91% of Canadians own a cell phone. Of these people, 42% have a Smartphone. $\to P(S|C)$ Determine, to the nearest percent, the probability that any Canadian you met during the month in which the survey was conducted would have a Smartphone.

$P(C) = 0.91$

$P(S|C) = 0.42$

$P(S \cap C) = ?$

$$\begin{aligned} P(S \cap C) &= P(C) \cdot P(S|C) \\ &= 0.91 \cdot 0.42 \\ &= \underline{\underline{0.3822}} \end{aligned}$$

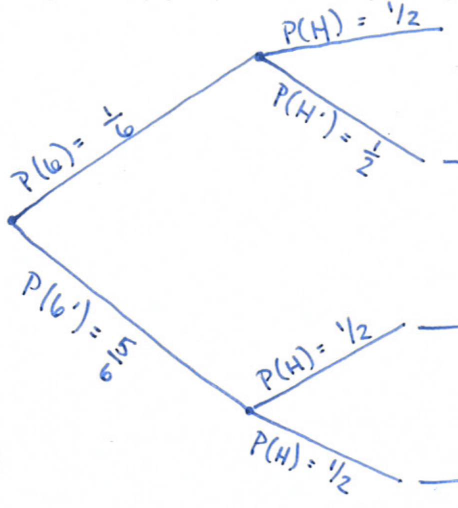
EX: Anne and Chantelle are playing a die and coin game. Each turn consists of rolling a regular die and tossing a coin. Points are awarded for rolling a 6 on the die and/or tossing heads with the coin:

- 1 point for either outcome INDEP → tree!
- 3 points for both outcomes
- 0 points for neither outcome

Players alternate turns. The first player who gets 10 points wins.

a) Determine the probability that Anne will get 1, 3, or 0 points on her first turn.

b) Verify your results for part a). Explain what you did.



→ $P(6 \cap H) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$ (3points)

→ $P(6 \cap H') = \frac{1}{12}$ (1 point)

→ $P(6' \cap H) = \frac{5}{12}$ (1 point)

→ $P(6' \cap H') = \frac{5}{12}$ (0 points)

$P(3) = \frac{1}{12}$

$P(1) = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$

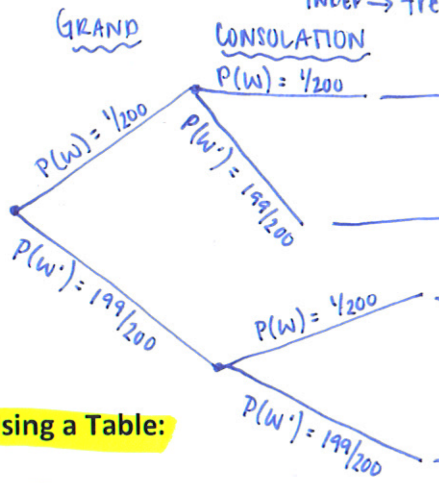
$P(0) = \frac{5}{12}$

add up to 1 ∴ verified!

EX: All 1000 tickets for a charity raffle have been sold and placed in a drum. There will be two draws. The first draw will be for the grand prize, and the second draw will be for the consolation prize. After each draw, the winning ticket will be returned to the drum so that it might be drawn again. Max has bought five tickets. Determine the probability, to a tenth of a percent, that he will win at least one prize.

INDEP → tree

↳ WL, WW, LW



→ $P(W \cap W) = \frac{1}{200} \cdot \frac{1}{200} = \frac{1}{40{,}000}$ ⇒ WW

→ $P(W \cap W') = \frac{1}{200} \cdot \frac{199}{200} = \frac{199}{40{,}000}$ ⇒ WL

→ $P(W' \cap W) = \frac{199}{200} \cdot \frac{1}{200} = \frac{199}{40{,}000}$ ⇒ LW

Add together!

$= \frac{399}{40{,}000}$

→ $P(W' \cap W') = \frac{199}{200} \cdot \frac{199}{200} = \frac{39601}{40{,}000}$ ⇒ LL

Using a Table:

And vs. Or:

EX: The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

a) Has a red car *and* got a speeding ticket

- We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is:

$$\frac{\mathbf{15}}{\mathbf{655}} \approx \mathbf{0.0226}$$

- Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

b) Has a red car *or* got a speeding ticket.

- We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people.
- So the probability is

$$\frac{\mathbf{195}}{\mathbf{665}} \approx \mathbf{0.2932}$$

- We also could have found this probability by:

$$P(had\ a\ red\ car) + P(got\ a\ speeding\ ticket) - P(had\ a\ red\ car\ and\ got\ a\ speeding\ ticket)$$

$$\frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{\mathbf{195}}{\mathbf{665}}$$

Conditional (Dependent) Probability:

EX: The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

a) Has a speeding ticket *given* they have a red car

- Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(ticket \mid red\ car) = \frac{15}{150} = \frac{1}{10} = \mathbf{0.1}$$

b) Has a red car *given* they have a speeding ticket

- Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(red\ car \mid ticket) = \frac{15}{60} = \frac{1}{4} = \mathbf{0.25}$$

IMPORTANT: Notice from the last example that $P(B \mid A)$ is ***not*** equal to $P(A \mid B)$.

Homework- 6B

Unit 3- Counting Methods & Probability

Lesson 7- Factorials & Permutations

Fundamental Counting Principle: Involves displaying the number of factors in determining an outcome as blanks. Then, the number of options for each factor are placed in each of the blanks. Numbers are then multiplied to get our final number of choices.

EX: There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

- There are 21 choices from the first category
- There are 18 choices from the second category

$$21 \cdot 18 = \boldsymbol{378\ choices}$$

EX: Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choice exactly one item from each category for your meal, how many different meal options do you have?

- There are 3 choices for an appetizer
- There are 5 choices for the main course
- There are 2 choices for the dessert

$$3 \cdot 5 \cdot 2 = \boldsymbol{30\ options}$$

EX: A quiz consists of 3 true or false questions. In how many ways can a student answer the quiz?

- There are 2 choices for the first, second and third question

$$2 \cdot 2 \cdot 2 = 2^3 = \boldsymbol{8\ ways}$$

EX: Many video games over the years have incorporated some version of a three part key sequence to unlock some portion of the game. For example: If there are three chains that have to be pulled in a certain order to unlock the door, how many different “combinations” are there? Prove your results with a tree diagram.

EX: Licence plates in a certain country consist of 6 characters. The first 4 can be any consonant and the last two have to be different numbers. How many licence plates can be made under this system?

EX: The paper on which you wrote your new friend's phone number was torn and the last two digits were lost. You do remember that the digits were not the same. How many 10 digit phone numbers might you have to try to guarantee you find the right one?

EX: In how many ways can a teacher seat four girls and three boys in a row of seven seats if

a) A boy must be seated at each end of the row?

b) There are no restrictions?

EX: Complete the following:

a) Cam is determining how many odd five digit numbers there are. Zero cannot be the first number (for example 02643 is NOT a number).

b) How many odd or even five digits number are there. Repeating is allowed.

EX: How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

- This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the same category (the category is: the letters of the word MATH) and each time we choose an item ***we do not replace it***, so there is one fewer choice at the next stage:
    - We have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T).
      Thus there are $\mathbf{4 \cdot 3 \cdot 2 \cdot 1 = 24}$ ways to spell a code worth with the letters MATH.

EX: Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

- Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze
  $\mathbf{8 \cdot 7 \cdot 6 = 336}$ ways the three medals can be awarded to the 8 runners.

NOTE: in these preceding examples, letters and the Olympic medals were awarded ***without replacement***; that is, once we have chosen a letter or winner of the gold medal, they are not eligible for the other slots or medal. ***Thus, at each succeeding stage of the solution there is one fewer choice*** (4, then 3, then 2, then 1 in the first example; 8, then 7, then 6 in the second).

- Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — ***for each question on the test***.

NOTE: the order of ***selection was important*** in each example: for the word, being placed first means that the word starts with that specific letter; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze.
Diving into the Olympics example, if we had chosen the same three people in a different order there might have been a different gold medalist.

- (Contrast this with the situation where we might draw three names out of a hat to each receive a participation ribbon; in this case the order of selection is ***not*** important since each of the three people receive the same prize. Situations where the order is ***not*** important will be discussed in the next section.)

EX: How many different four letter arrangements can you rearrange from the word **QUESTION**.

a) Any letter can be in any position

b) The first and the last letters must be consonants

c) The first letter must be **T**

d) The word must contain **N**

Note:

- Anytime you have to distinguish components of a question with the word **and,** that represents **multiplication** between the digits.
- Anytime you have to distinguish components of a question with the word **or**, that represents **addition** between the digits.
- Anytime there is a **restriction**, deal with that first.

Homework

Lesson 7 Con't- Factorials & Permutations

Factorial Notation:

Whenever the numbers descend from the largest number to 1, while multiplying them together, you have something that can be rewritten in factorial notation.

- For example:
    - The fundamental counting principle yields the result $6 \times 5 \times 4 \times 3 \times 2 \times 1$.
        - Such a product can be abbreviated as $6!$
        - So, 5! would be $5 \times 4 \times 3 \times 2 \times 1$
    - In general, $n! = n(n-1)(n-2)...(3)(2)(1)$, where $n \in N$
        - Notice that they descend by 1 every time.

Factorials are numbers and as such, they can be multiplied, divided, added, factored etc.

- When simplifying factorial expressions, always start with the **largest factorial** and "march" the terms down until you get to the smaller factorial.

EX: a) $\frac{7!}{3!}$

b) $\frac{(n+2)!}{n!}$

c) $\frac{(n-1)!}{(n+2)!}$

d) $\frac{(n!)}{n(n-1)!}$

Whenever solving an equation, ensure that the answers are **natural numbers** (greater than 1).

EX: $\frac{n!}{(n-2)!} = 42$

EX: $\frac{(n+4)!}{(n+2)!} = 6$

Permutation:

The arrangement of objects or people in a line is called a linear **permutation**. ***In a permutation, the order of the objects is important***.

- When the objects are distinguishable or different from one another, a new order of objects creates a new permutation.
- EX: Seven different objects can be arranged in 7! ways.
  - $7! = 7\times6\times5\times4\times3\times2\times1$
- EX: If there are seven members on the student council, in how many ways can the council select three students to be the chair, the secretary, and the treasurer of the council?
  - Using the fundamental counting principle, there are (7)(6)(5) possible ways to fill the three positions.
  - Using the factorial notation, to get the (7)(6)(5), we have to get rid of the (4)(3)(2)(1) or 4! To do this, we divide.
  - $\frac{7!}{4!} = \frac{7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}$ = (7)(6)(5) = 210
- The notation, *nPr* , is used to represent the number of permutations, or arrangements in a definite order, of *r* items taken from a set of *n* distinct items.
  - This is used when **order matters AND the choices are *not* being replaced.**

A formula for ${}_n\mathrm{P}_r$ is ${}_n\mathrm{P}_r = \frac{n!}{(n-r)!}$, $n \in N$ (on your formula sheet)

- Using permutation notation,
  - ${}_7\mathrm{P}_3$ represents the number of arrangements of **three** objects taken from a set of **seven** objects.
  - ${}_7\mathrm{P}_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 7\times6\times5 = 210$

EX: Evaluate ${}_8\mathrm{P}_5$ using factorial notation. Check with your calculator.

EX: Solve for n if ${}_{n}P_{2} = 56$.

EX: I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

- Since we are choosing 4 paintings out of 9 **without replacement** where the **order of selection is important** there are $\mathbf{{}_{9}P_{4}} = \boldsymbol{9 \cdot 8 \cdot 7 \cdot 6 = 3024\, permutations}$

EX: Use permutations to determine how many arrangements there are of

a) two letters from the word **SYLVAN**

b) three letters from the word **PRODUCTS**

c) five letters from the word **FORMULA**

EX: Vehicle license plates consist of 3 different letters, followed by 4 different digits. If the letters I, O, Y, and Z are not used, determine the number of different licenses plates they can make.

a) Use FCP to find the answer

b) Use Permutations to find the answer

Note: 0! Is NOT 0, it equals 1.

HELPFUL HINTS:

- When to use a PERM:
    - Basic "No repetition, no doubles, all different" questions.
    - Once it's placed it's gone
- When to use FCP:
    - When repetitions are allowed, or when there are restrictions
    - Easier to use when there's a lot of restrictions then PERM
- SPECIFIC TYPES OF QUESTIONS:
    - WORDS:
        - Like scrabble- once it's placed it's gone- PERM or FCP
    - Licence plates, phone numbers, etc
        - Watch for restrictions, BUT if it does not state a "no repeating" restriction, assume you CAN repeat numbers- FCP ONLY
        - If no repetitions are allowed, you are placing the option and then it is no longer available- PERM or FCP
    - Sitting in a row, groups with designations, etc
        - Watch for "placements" (chairman, spokesperson) or specific positions within the line (Billy needs to be first in line)- this means that when the person is picked and placed, they are no longer available- PERM or FCP

- Anytime something is "placed into a position" (letters or specific numbers), it can be helpful to place a "hypothetical" letter or number in that position, then count how many options you have left based on **if repetitions are allowed or not** to address the next blank.
    - Look back at the "Cam" example or the "QUESTION" example

Homework

Lesson 8- Identical Objects

Restrictions:
Just like with FCP, deal with restrictions first, then complete the rest of the equation. I recommend using the FCP blanks to help set out where the restrictions are located.

EX: How many arrangements are there with the word **FRIDAY** when the vowels stay together and at the beginning?

EX: How many arrangements are there with the word **ORANGES** if:

a) The letters **G, E,** and **S** must stay together, but in no particular order?

b) The consonants must stay together?

Repetition:
Whenever you are considering arrangements of items where some of them are identical, it makes sense that there will be fewer possible arrangements because some of the arrangements will be the same.

- Consider the word pool. We can arrange all of the letter differently like this

pool opol oopl oolp polo oplo olpo olop ploo lpoo lopo loop
pool opol oopl oolp polo oplo olpo olop ploo lpoo lopo loop

- In general, a set of *n* objects with *a* of one kind that are identical, *b* of a second kind that are identical, and *c* of a third kind that are identical, and so on, can be arranged in $\frac{n!}{a!b!c!}$ different ways.

EX: How many different "words" can be formed by the word "HAZLETT"? Solve by expanding the factorial and cancelling.

- Total number of letters is 7
- The "T's" repeat 2 times

$$\frac{7!}{2!}=\frac{7\cdot6\cdot5\cdot4\cdot3\cdot\cancel{2\cdot1}}{\cancel{2\cdot1}}=\mathbf{2520}$$

EX: How many different words can be formed by RED DEER? Solve by expanding the factorial and cancelling.

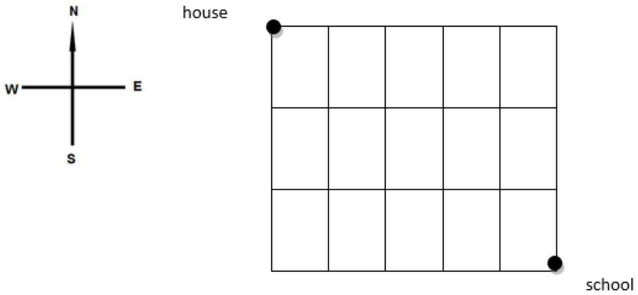
- Total number of letters is 7
- The "R's" repeat 2 times
- The "D's" repeat 2 times
- The "E's" repeat 3 times

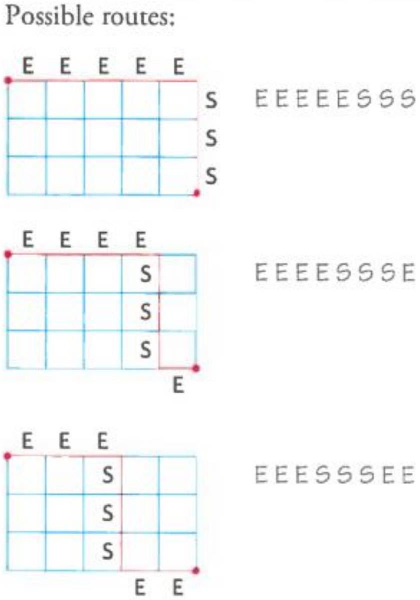
$$\frac{7!}{2!\,2!\,3!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{\cancel{(2\cdot 1)(2\cdot 1)(3\cdot 2\cdot 1)}} = 7\cdot 6\cdot 5 = \mathbf{210}$$

EX: How many different eight-letter arrangements can you make using the letters of aardvark?

EX: How many different arraignments are there using all of the letters of Mississippi?

EX: Julie’s home is three blocks north and five blocks west of here school. How many routes can Julie take from home to school if she always travels either south or east?

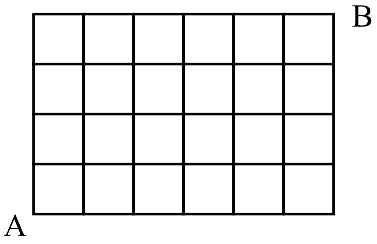




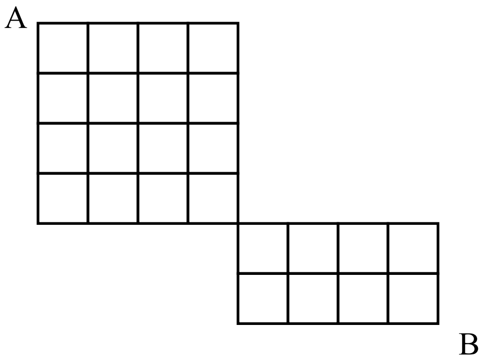
- No matter what way she goes, she will walk east 5 blocks and south 3 blocks. So, in total, she will walk 8 blocks.
- All the east blocks are identical, and all the south blocks are identical. If you start writing out the possible routes, you will see that this is just an identical objects problem. She will walk 8 total blocks with east repeating 5 times and south repeating 3 times

$$\frac{\mathbf{8!}}{\mathbf{5!\,3!}} = \mathbf{56\ \mathit{ways}}$$

EX: Consider a town with streets and avenues at right angles to each other. How many paths can you follow from one corner of the town to the other, from A to B, if you always move closer to your destination?



EX: How many ways are there from A to B – assume you always move closer to your destination.



EX: How many ways can one French poster, two Mathematics posters, and three Science posters be arranged in a row on a wall if

a) The two mathematics posters must be together on the end?

b) The three science posters must be together?

c) The three science posters cannot all be together?

EX: How many 4-digit odd numbers can you make using the digits 1 to 7 if the numbers must be less than 6000 and no digits are repeated?

EX: How many even 5 digit numbers are there? Remember, 01234 is 1234 and therefore is NOT considered 5 digits. No repeating of digits.

Homework

Lesson 9- Combinations

Combinations:

A **combination** is a selection of a group of objects, taken from a larger group, for which the kinds of objects selected is important, ***but not the order*** in which they are selected.

- The notation ${}_nC_r$, or $\begin{pmatrix} n \\ r \end{pmatrix}$, represents the number of combinations of *n* items taken *r* at a time, where n ≥ r and r ≥ 0.

$$\begin{aligned} {}_nC_r &= \frac{{}_nP_r}{r!} \\ &= \frac{\frac{n!}{(n-r)!}}{r!} \\ &= \frac{n!}{(n-r)!r!} \end{aligned}$$

    - This is used when **order does not matter BUT the choices are *not* being replaced.**

In all of these questions, we can still use the fundamental counting principle to help our calculations.

EX: A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

- Since we are choosing 4 people out of 35 **without replacement** where the order of selection is **not important** there are

$${}_{35}C_4 = \frac{35 \cdot 34 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1}$$ = **52,360 combinations.**

EX: The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

- In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats.
    - There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans ***and***
    - There are ${}_{14}C_9 = 2002$ ways to choose the 9 Democrats.
- But now what? How do we finish the problem?
    - There must be 3003 · 2002 = **6,012,006 possible ways of selecting the members of the Defense Subcommittee.**

EX: In a group of 9 students, there are 4 females and 5 males.

a) How many different 4-member committees selected from this group have 2 females ***and*** 2 males?

$$\binom{4}{2} \cdot \binom{5}{2} = \mathbf{60}$$

b) How many different 2-member ***or*** 3-member committees can be formed from this group of students?

$$\binom{9}{2} + \binom{9}{3} = \mathbf{120}$$

EX: There are 12 females and 18 males in a grade 12 class. The principal wishes to meet with a group of 5 students to discuss graduation.

a) How many selections are possible?

b) How many selections are possible if the group consists of two females and three males?

c) One of the female students is named Brooklyn. How many five-member selections consisting of Brooklyn, one other female, and three males are possible?

As with permutations, combinations can have restriction. Watch out for the words ***"at most"*** or **"*at least"*** are used.

EX: Josie is writing a math exam. The instructions say that she must answer a specified number of questions from each section. How many different selections of questions are possible if:

a) She must answer two of the four questions in part A and three of the five questions in part B?

b) She must answer two of the four questions in part A and at least four of the five questions in part B?

Note: the use of the word *"at least"* triggers different possibilities.

- "at least 4 of the 5" means:
    - 4 of the 5 OR 5 of the 5
- In these cases, the word *"or"* implies **addition**.
- The exact same thing happens with the words "at most". Many such questions involve committees or cards.

EX: A standard deck of 52 cards has the following characteristics:

- 4 suits (spades, clubs, diamonds, and hearts).
- Each suit has 13 cards.
- Two suits are black (spades and clubs)
- Two suits are red (diamonds and hearts)
- There are 4 of each numbered card (4 kings, 4 eights, etc.)
- Face cards are considered to be Jacks, Queens, and Kings.

In a card game, five cards are dealt (given) to each player – this is called the player's hand.

a) In how many ways can one such "hand" be dealt with 3 black cards and 2 reds?

b) In how many ways can a hand be dealt with at least 3 black cards.

c) In how many ways can a hand be dealt with at most 3 kings.

EX: At the end of a volleyball camp, each player shakes every other player hand, only once. If there were 276 handshakes in all, how many players attended the camp?

Homework

Lesson 10- Permutations & Combination Questions

This is a practice lesson- please proceed to My Open Math and complete the homework.

Tips:

- Handle **restrictions** first.
- **Permutations**- you are selecting from a group without replacement AND order matters
- **Repetition**- be sure to divide by any repetition!
- **Combination**- you are selecting from a group without replacement BUT order does NOT matter.

EX: A coach for a co-ed volleyball team wants a team picture taken. There are five boys and three girls on the team. The photographer wants the boys to sit together and the girls to sit together, with the two tallest students, Jill and Jack to sit on the outside ends and the coach to sit between the two groups. How many ways can the team and coach be organized for this picture?

EX: A teacher is creating a test for tomorrow's class. She has 11 questions she wishes to use. Four of the questions are to go in the first section, two of the remaining questions go in the second sections, three in the next and two in the last. How many ways can the 11 questions be places in the four sections of the exam?

EX: A kindergarten class has 20 students, which include Bess and Jess. How many ways could the students be seated in a row where Bess and Jess are NOT together?

EX: In Cribbage, the dealer deals 6 cards to each player from a standard deck of 52 cards. In how many ways could one player's hand contain *at least* 5 hearts if she was dealt all of her cards first.

Homework

Lesson 11- Probability using Permutations & Combinations

Probability:

Like we've previously learnt, when dealing with probability, you create a fraction:

$$\frac{favorable\ outcome}{total\ number\ of\ outcomes}$$

We will be doing just this, but the fraction numerator and denominator come from a FCP, permutation or combination.

EX: A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

- There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9),
  - So there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.
- To have no repeated digits, all four digits would have to be different, which is selecting without replacement.
  - We could either compute $10 \cdot 9 \cdot 8 \cdot 7$,
  - OR notice that this is the same as the permutation ${}_{10}P_4 = 5040$.
- The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers.

This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$$

EX: In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

- In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine.
  - Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$.
  - Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_{6}C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815$$

EX: In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

- As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$.
- In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers.
  - The number of ways to choose 5 out of the 6 winning numbers is given by ${}_{6}C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$.

  - Thus the number of favorable outcomes is then given by the Basic Counting Rule: $_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$.
- So the probability of winning the second prize is.

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205$$

EX: A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

EX: Daisy is putting up tiles on a billboard sign. She wants to make the word WELCOMING. If she picks 3 of these tiles at random, what is the probability that two are consonants and one is a vowel?

EX: Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

- In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated.
  - Thus we use combinations to compute the possible number of 5-card hands, $_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.
- For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck.
  - Since there are four Aces and we want exactly one of them, there will be $_4C_1$ ways to select one Ace
  - Since there are 48 non-Aces and we want 4 of them, there will be $_{48}C_4$ ways to select the four non-Aces.
  - Now we use the Basic Counting Rule to calculate that there will be $_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.
- Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

EX: Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

- The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

EX: Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

EX: In a group of 11 students, there are 5 females and 6 males

- How many different 4-member committees are possible there must be at least 1 girl?
    - Subtract the case where there are no girls from the total.
        - You could also add up all the cases that have 1 or more girls- it just takes a bit longer.

$$\binom{11}{4} - \left[\binom{5}{0} \cdot \binom{6}{4}\right] = 315$$

- Determine the probability that a 4-member committee chosen at random from this group will consist of at least 1 female. Write your answer as a percent.

$$\frac{\binom{11}{4} - \left[\binom{5}{0} \cdot \binom{6}{4}\right]}{\binom{11}{4}} \cdot 100\% = 95\%$$

EX: Kelsey needs to create a four-digit pass code to access her voicemail. She can repeat some of the digits, but all four digits cannot be the same.

a) Determine the probability that her pass code will be greater than 5000

b) Determine the probability that the first number and the last number of her pass code is 4

c) Determine the probability that the first digit of her pass code will be odd and the last digit will be even.

Mutually Exclusive:

Recall, for an even to be mutually exclusive, the events do not impact each other/ they do not overlap in a Venn Diagram.

EX: A database reports that 56% of the cars on a dealership lot have heated leather seats, 49% have a sunroof, and 27% have neither.

a) Are the events mutually exclusive? Prove it using the formula for mutually exclusive events.

b) In non-mutually exclusive, what is the probability that a car at the dealership has heated leather seats and a sunroof?

c) What is the probability of a car having a sunroof but no heated leather seats?

Conditional Probability:

Recall when events are independent vs dependent.

- Independent: is when the outcome of one event does NOT impact the outcome of another event.
- Dependent: is when the outcome of one event does impact the outcome of another event.

EX: Kyler has four loonies, three toonies, and five quarters in his pocket. He needs two loonies for a parking meter. He reaches into his pocket and pulls out two coins at random. Determine the probability that both coins are loonies.

EX: Jeremiah is going on a cruise up the Nile. According to the travel brochure, the probability that he will see a camel is $\frac{4}{5}$, and the probability that he will see an ibis is $\frac{3}{4}$. Determine the probability that Jeremiah will see the following:

a) A camel and an ibis

b) Neither a camel nor an ibis

c) Only one of these sights

Homework

Unit 4- Rational Expressions

Lesson 12- Factoring Review

Factoring will be an essential skill for this unit. Please make sure you are confident with the following skills.

Factor the Greatest Common Factor from a Polynomial

- **Distributive Property:** if a, b, c are real numbers, then:
  - $a(b+c) = ab + ac$ and $ab + ac = a(b+c)$
  - The form on the left is used to **multiply**. The form on the right is used to **factor**.

EX: Factor the following polynomial. Identify the factors

$$4x + 2$$

- The GCF (greatest common factor) of both terms is 2
  - Write down the GCF

$$2$$

  - Start a bracket, and then write what you have left after you divide each term by 2

$$2(2x+1)$$

  - **The factors are 2 and $(2x+1)$**

EX: Factor the following polynomial. Identify the factors

$$6x^2y^3 + 4x^3y^6 + 2x^2y^2$$

- The GCF (greatest common factor) of both terms is

$$2x^2y^2$$

$$2x^2y^2(3y + 2xy^4 + 1)$$

  - **The factors are $(2x^2y^2)$ and $(3y + 2xy^4 + 1)$**

EX: Factor the following:

1. $12x^3 + 9yx^2$

2. $15m^3n + 20mn^2 - 10m^2n$

3. $6x^2y^2 + 8x^2y - 12x^6y - 4xy^3$

Trinomial Factions- When Leading Coefficient is 1:

Using the general form of $ax^2 + bx + c$, ensure that $a = 1$.

- If this is the result, then the factors are a result from the numbers that multiply to get c and add to get b.
- For example:

$$x^2 - 2x - 3$$
$$(-3)(1) = -3$$
$$(-3) + (1) = -2$$
$$(x - 3)(x + 1)$$

EX: Factor the $h^2 + 4h - 5$

- Find the numbers that multiply to -5

$$-5 \text{ } and \text{ } 1 \qquad\qquad 5 \text{ } and - 1$$

- These numbers also need to add up to $+4$

$$5 \text{ } and - 1$$

- Therefore, my factors are the following:

$$\boldsymbol{(x + 5)(x - 1)}$$

EX: Factor the following:

1. $x^2 + x - 6$
2. $x^2 - 9x + 20$
3. $x^2 + 7x + 6$
4. $2x^2 + 7x - 15$

Trinomial Factions- When Leading Coefficient is NOT 1:

Using the general form of $ax^2 + bx + c$, ensure that $a \neq 1$.

- For example:

$$6x^2 + 11x + 4$$

What multiplies to $(6)(4) = 24$

What adds to 11

$$(3)(8) = 24$$

$$(3) + (8) = 11$$

THESE ARE NOT YOUR FACTORS THOUGH!!

Replace $\mathbf{11x\ with\ 3x\ and\ 8x}$

$$6x^2 + 3x + 8x + 4$$

$$3x(2x + 1) + 4(2x + 1)$$

$$(3x + 4)(2x + 1)$$

EX: Factor the following:

1. $2x^2 + 13x + 20$
2. $2t^2 - t - 6$
3. $4x^3 - 14x^2 + 6x$
4. $4x^2 - 8x + 3$

Factor Difference of Squares

- **Product of conjugates pattern:** if a and b are real numbers

$$(a-b)(a+b) = a^2 - b^2$$

The product is called a difference of squares.

EX: Factor $4x^2 - 25$

- My a value would be $2x$
- My b value would be 5
- Putting it into the above formula, you get:

$$\boldsymbol{(2x-5)(2x+5)}$$

EX: Factor the following:

1. $16x^2 - 36$

2. $9x^2 - 81y^2$

3. $3x^2 - 75y^2$

Homework

Lesson 13- Simplifying Rational Expressions

Rational Functions:

A rational function is a function which takes the form $f(x) = \frac{n(x)}{d(x)}$ where $n(x)$ and $d(x)$ are polynomial functions and $d(x) \neq 0$.

Non-Permissible Values:

- It is very important to remember that the denominator of any fraction cannot equal zero. When dealing with rational functions, you must equate the denominator to zero and solve for the values that are denoted NPV's (these values are also important for the domain- restrictions) This may require you to factor and then solve.
    - The numerator CAN but 0, just the denominator cannot be.
    - If the denominator does equal zero, the rational expression is undefined.

Determining the Values for the NPV's:

1. Factor the DENOMINATOR (if possible)
2. Set each factor in the denominator equal to zero (if not factored, then just set the whole denominator equal to zero)
3. Solve the equation and for there you have the value that x CANNOT equal.

Here are some examples of rational expressions:

$$-\frac{13}{42} \qquad \frac{7y}{8z} \qquad \frac{5x+2}{x^2-7} \qquad \frac{4x^2+3x-1}{2x-8}$$

EX: What are the non-permissible values for the expression $\frac{4x^2-25}{2x-3}$

- We want to know what x would need to be in order to make the denominator 0. So we will make the expression

$$2x - 3 = 0$$

- Solve

$$\boldsymbol{x = \frac{3}{2}}$$

- This means that $x \neq \frac{3}{2}$ and that we have an NPV value of $\frac{3}{2}$

EX: State the NPV's for the following expressions:

1. $\frac{17x-2}{8}$

2. $\frac{6x^3-5x}{12x}$

3. $\frac{6x^3-5x}{12x^2}$

4. $\frac{9}{16+z^2}$

5. $\frac{2x-4}{-6x+2}$

6. $\frac{18q+2}{7q^2-q}$

7. $\frac{-3m}{m^2-64}$

8. $\frac{17}{2k^2+10}$

9. $\frac{(x+5)(2x-3)}{(2x+3)(x+5)}$

10. $\frac{x-4}{x^2-5x-6}$

11. $\frac{y+2}{8y^2-14y+5}$

NOTE: Non-permissible values lead to your restrictions! The restriction would be when the variable **CANNOT** equal the NPV value. Go back and state the restriction for a few of the examples above.

EX: In each case, write a rational expression with the given variable and non-permissible value:

1. Variable is x, NPV is 3

2. Variable is a, NPV is 0 and 4

3. Variable is p, NPV is 1/3

4. Variable is m, NPV is -8

Simplify Rational Expressions:

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

Steps for Simplifying Rational Expressions:

1. Factor the numerator and denominator completely.
2. Grab your NPV's **before you do step 3.**
3. Cross off (cancel) common factors.
4. Write what remains.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

EX: Solve and simplify: $\frac{x^2-x-2}{x^2-3x+2}$

- First we factor both the numerator and the denominator

$$\frac{(x-2)(x+1)}{(x-2)(x-1)}$$

- Then we cancel out any common factors

$$\frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x-1)}$$

- Leaving our answer as

$$\boldsymbol{\frac{(x+1)}{(x-1)}}$$

EX: Simplify the following rational expressions and state the restrictions.

1. $\frac{16x^2}{50xy}$

2. $\frac{10x(x+2)(x-2)}{15x^2(x-2)}$

3. $\frac{6x+3}{6x}$

4. $\frac{x+2}{4-x^2}$

5. $\frac{12p^2-8p}{6p^3-4p^2}$

6. $\frac{3z^2+9z}{12z+36}$

7. $\frac{2x^2-7x+3}{x^2-2x-3}$

EX: Write a rational expression that is equivalent to:

1. $\frac{8}{12}$

2. $\frac{4x^2+8x}{4x}$

EX: Determine if the following rational expression are equivalent:

1. $\frac{9}{3x-1}$ $\frac{-18}{2-6x}$

2. $\frac{2-2x}{4x}$ $\frac{x-1}{2x}$

Homework

Lesson 14- Multiply & Divide Rational Expressions

Multiply Rational Expressions

To multiply rational expressions, we multiply the numerators and multiply the denominators. Then, if there are any common factors, we cross them off to simplify.

- If p, q, r, s are polynomials where $q \neq 0$ and $s \neq 0$, then:

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded (aka- we will be always excluding NPV's). We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero.

How to Solve Rational Functions that are being Multiplied:

1. Factor each numerator and denominator completely.
2. Multiply the numerator and denominators together (write all the factors side by side)
3. Cross out common factors.

EX: Multiply and state the non-permissible values $\frac{5x}{x^2+5x+6} \times \frac{x^2-9}{6x^2}$

- First we factor

$$\frac{5x}{(x+2)(x+3)} \times \frac{(x-3)(x+3)}{6x^2}$$

- We can think of this expression more simply as

$$\frac{5x(x-3)(x+3)}{6x^2(x+2)(x+3)}$$

- You take your NPV's from this stage, BEFORE you cancel out common factors. Therefore your NPV's would come from $6x^2, (x+2), (x+3)$
    - Set each factor equal to zero and solve for x:
        - $6x^2 = 0 \qquad x = 0$
        - $x + 2 = 0 \qquad x = -2$
        - $x + 3 = 0 \qquad x = -3$
            - Therefore, $x \neq 0, -2, -3$
- Then cancel any common factors

$$\frac{5\cancel{x}(x-3)\cancel{(x+3)}}{6x^{\cancel{2}}(x+2)\cancel{(x+3)}}$$

- This is now simplified to

$$\boldsymbol{\frac{5(x-3)}{6x(x+2)}}$$

EX: Simplify the following, state any restrictions.

1. $\frac{4x-16}{14x-7} \times \frac{2x-1}{(x-4)^2}$

2. $\frac{2x^3-4x^2}{3x^2-9x} \times \frac{x^2}{x^2-4}$

3. $\frac{100x^5y}{27y^4} \times \frac{81xy^5}{250x^9}$

4. $\left(\frac{14x+2}{8x-80}\right)\left(\frac{8}{70x+10}\right)$

5. $\frac{6v-42}{40v^2+72v} \times \frac{(5v+9)(3v+9)}{18v+54}$

6. $\frac{6a+20}{2a-4} \times \frac{8a-32}{2a+14} \times \frac{2a^3-4a^2}{(a-4)(3a+10)}$

EX: With the appropriate restrictions, the product of $\left(\frac{12x-24}{3x^2-12}\right)\left(\frac{6x+12}{2}\right)$ reduces to a whole number, *k*, the value of *k* is _____. (REMEMBER- always fill in numerical responses from **left to right** and you do not have to fill in all the boxes)

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We "flip" the fraction.

- If p, q, r, s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then:

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

How to Solve Rational Functions that are being Divided:

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerator and denominators together (write all the factors side by side)
4. Cross out common factors.

EX: Divide and state any non-permissible values $\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$

- Rewrite as a multiplication statement

$$\frac{3n^2}{n^2-4n} \times \frac{n^2-7n+10}{9n^2-45n}$$

- Then we factor

$$\frac{3n^2}{n(n-4)} \times \frac{(n-5)(n-2)}{9n(n-5)}$$

- Multiply without expanding, then cancel any common factors

$$\frac{\cancel{3n^2(n-5)}(n-2)}{\cancel{9n^2}(n-4)\cancel{(n-5)}}$$

- This is now simplified to

$$\boldsymbol{\frac{(n-2)}{3(n-4)}}$$

- Careful with your NPV's when dividing. The numerator and denominator in the second term cannot equal zero, along with the denominator of the first term. Therefore the following are all the NPV's we need to consider: $n, (n-4), (n-5), (n-2), 9n, (n-5)$
  - Set each factor equal to 0 and solve for x:
    - $n = 0$ $\quad n = 0$
    - $n - 4 = 0$ $\quad n = 4$
    - $n - 5 = 0$ $\quad n = 5$
    - $n - 2 = 0$ $\quad n = 2$
    - $9n = 0$ $\quad n = 0$
      - Therefore, $x \neq 0, 2, 4, 5$
        - You only have to write the NPV value once, even if it occurs multiple times.

EX: Simplify the following. State any restrictions.

1. $\frac{9}{2n} \div \frac{7n}{8n^2}$

2. $\frac{(x+1)}{(x^2+x-6)} \div \frac{(2x+2)}{(x^2+3x)}$

3. $\frac{56m^3+32m^2}{40m^2+16m} \div \frac{35m+20}{40m^2+16m}$

4. $\frac{21k^2-14k-7}{15k^2-10k-5} \div \frac{7k^2-56k+49}{40k^2-40k}$

5. $\dfrac{\frac{4}{x}-\frac{3}{x}}{\frac{16}{x^2}}$

6. $\frac{a-6}{3a^2+9a} \div a+3$

EX: Write two rational expressions whose:

1. Produce is $\frac{x+1}{5x}, x \neq 0, 3$
2. Quotient is $\frac{3x^2}{x+2}, x \neq -2, 3$

Homework

Lesson 15- Adding & Subtracting Rational Expressions

Add & Subtract Rational Expressions with a Common Denominator

You check if they have a common denominator. If they do, you add or subtract the numerators and place the sum or difference over the common denominator.

- If $p, q, and\ r$ are polynomials where $r \neq 0$, then
$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$$
- If $p, q, and\ r$ are polynomials where $r \neq 0$, then
$$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

EX: Add the following rational expressions: $\frac{3y}{4y-3}+\frac{7}{4y-3}$

- The fractions have a common denominator, so simply add the numerators (collect like terms if possible) and place the sum over the common denominator.
$$\boldsymbol{\frac{3y+7}{4y-3}}$$
- The numerator and denominator cannot be factored; therefore, this fraction is simplified.

EX: Subtract the following rational expressions: $\frac{n^2}{n-10}-\frac{100}{n-10}$

- The fractions have a common denominator, so simply subtract the numerators (collect like terms if possible) and place the sum over the common denominator.
$$\frac{n^2-100}{n-10}$$
- Factor the numerator.
$$\frac{(n-10)(n+10)}{n-10}$$
- Simplify by cancelling out common factors.
$$\frac{\cancel{(n-10)}(n+10)}{\cancel{n-10}}$$
- Therefore the final solution would be
$$\boldsymbol{n+10}$$
  - **NOTE:**
    - Always foil in the subtraction sign if the second numerator is anything more than a single term.
    - Remember, you would be taking you NPV's BEFORE you simply, but after you factor.

Add and Subtract Rational Expressions with Different Denominators

1. If your denominators are not equal, factor them fully first.
2. From the factored form, determine your LCD. (Every factor has to be accounted for ONCE)
    a. Your LCD has all you restrictions (NPV values!!)
3. Multiply each fraction by only the factors that are missing! This will NOT be the whole LCD.
4. Distribute (FOIL) the numerators, leave the denominator as factors.
5. Now that you have each fraction with a common denominator, you can add or subtract the numerators and write them on top of the LCD.
6. Simplify if possible.

EX: Simplify the following.

1. $\frac{11}{2x}+\frac{5}{2x}$

2. $\frac{5n}{6m^3}+\frac{3n}{2m}$

3. $\frac{3}{n-5}+\frac{5n}{n+2}$

4. $\frac{2x}{3x-9}+\frac{x+5}{12-4x}$

5. $\frac{x+2}{4}+\frac{3x}{x-5}$

EX: Subtract the following rationals $\frac{x}{x-3}-\frac{x-2}{x+3}$

- Determine the LCD of $x-3, x+3$

$$(x-3)(x+3)$$

- Determine what you would need to multiply each denominator by to get the LCD. Then multiply the numerator and denominator by that factor.

$$\left(\frac{x+3}{x+3}\right)\frac{x}{x-3}-\frac{x-2}{x+3}\left(\frac{x-3}{x-3}\right)$$

- Write your expression over the LCD of $(x-3)(x+3)$.

$$\frac{x(x+3)}{(x-3)(x+3)}-\frac{(x-2)(x-3)}{(x-3)(x+3)}$$

- Subtract your numerators together over the LCD (collect like terms at this point). **Be sure to FOIL in the negative sign to the second fractions numerator.**
  - ***Notice, we are foiling the numerators, but keeping the denominators in factored form.***

$$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$$
$$\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$$
$$\frac{8x-6}{(x-3)(x+3)}$$

- If you can factor anything out of the numerator, do so.

$$\boldsymbol{\frac{2(4x-3)}{(x-3)(x+3)}}$$

- If a factor could've been cancelled out, you would do that, but since there are no factors in common, this is the simplified solution.

EX: Simplify the following.

1. $\frac{3}{2x}-\frac{4}{3}$

2. $\frac{5m}{3m-1}-\frac{6m}{2m+1}$

3. $\frac{2s}{3s+6}-\frac{7}{4s+8}$

4. $\frac{9x}{3x-2}-\frac{6}{3x-2}$

5. $\frac{4x+1}{x^2-9}-\frac{9x-1}{x^2-2x-15}$

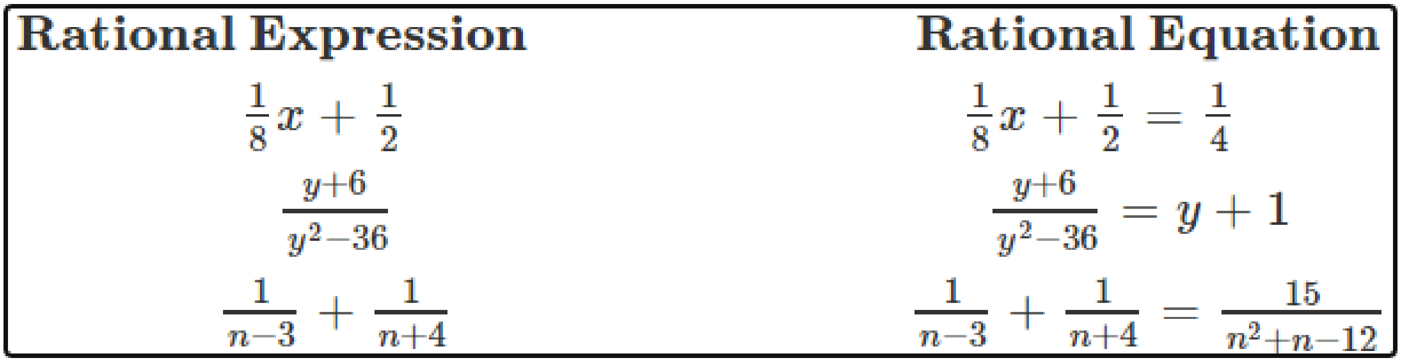
Homework

Lesson 16- Solving Rational Equations

Rational Equations

A rational equation is two rational expressions connected by an **equal sign**.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.



Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

How to Solve Rational Equations:

1. Factor denominators to find the LCD for the entire equation.
    a. The LCD will also provide you your NPV values!
2. Multiply EVERY numerator by the WHOLE LCD (this will allow us to cancel off the denominators).
    - **Notice, this is very different then our last lesson! We are trying to get rid of the fraction aspect.**
3. Once you have only have “numerators” left, solve the resulting equation.
    a. If there is just an x then **isolate** the variable on one side, all numbers to the other side.
    b. If there is an x^2 then bring everything to one side of the equation and **factor** to find the solutions. This could be basic factoring or using the quadratic formula.
4. Check your answers
    a. If any of your solutions are equal to the NPV’s you found in step one, then cross them off. They are what is called an **extraneous** solution because it causes the expression to be undefined.
        i. We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.
    b. Check the remaining numbers by subbing them back into the original equation. Ensure that the left side equals the right side.

EX: Solve the following $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$

- Determine the LCD and NPV's

$$LCD: 6x$$
$$Therefore\ 6x = 0$$
$$NPV: x \neq 0$$

- Multiply every numerator by the LCD

$$\frac{(6x)1}{x}+\frac{(6x)1}{3}=\frac{(6x)5}{6}$$

$$\frac{(6\cancel{x})1}{\cancel{x}}+\frac{(\cancel{6}x)1}{\cancel{3}}=\frac{(\cancel{6}x)5}{\cancel{6}}$$

$$(6)(1)+(2x)(1)=(x)(5)$$

$$6+2x=5x$$

$$6=3x$$

$$\boldsymbol{2=x}$$

- Check solution against the NPV, since they are not the same number, we are good to proceed into subbing in our solution into the original equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

$$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

$$\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

$$\boldsymbol{\frac{5}{6}=\frac{5}{6}}$$

✓

EX: Solve the following.

1. $\frac{1}{3}+\frac{1}{r}=\frac{1}{3r}$

2. $1-\frac{3}{5x+4}=\frac{1}{5x+4}$

3. $\frac{n-6}{n^2-3n}-\frac{n+6}{n^2-3n}=\frac{2}{n}$

4. $\frac{16}{x^2-4x}=\frac{4}{x-4}-\frac{1}{x}$

5. $\frac{5}{3x-1}+\frac{3x}{3x+1}=1$

6. $\frac{2}{k-6}=\frac{6k-12}{(k-6)(k-3)}+\frac{k-1}{k-3}$

7. $\frac{1}{x-4}+\frac{x}{x-2}=\frac{2}{x^2-6x+8}$

8. $\frac{1}{x-1} - \frac{2x}{x+1} = \frac{2}{x+1}$

9. $\frac{11}{2x^2+x-15} = \frac{5}{2x-5} - \frac{x}{x+3}$

Work Problems- Building the Equation:

STEPS:

- Once you determine the quantity that is unknown, designate a variable for it. (Usually this is what they are asking you to solve for)
- Break down every aspect of the question and write individual equations for each part
- Put together the parts of the equation with the appropriate sign (add, subtract, multiply, divide).
- Solve the equation
- Ensure the solution makes sense (no negatives for time, etc).

EX: Define a variable and write a rational expression to describe each situation.

a) The time it takes to travel 15 km when the speed has been decreased by 5 km/h.

b) The unit cost when the total cost is $4000 and the number of items has increased by 5.

c) The speed of a car that travelled 50 km in twice the time. The fraction of a job completed by Shannon in 1 min if it takes her 5 min longer than Brad to complete the job.

EX: John flies 308km in the same time Jessica flies 329km. Jessica flies, on average, 6 km/h faster then John. Let John's average speed be x km/h. Calculate Jessica's average speed and time taken for the trip. HINT- make a chart!

EX: Competing in an endurance race, Shannon cycled for 120km and then swam for 12km. Her average cycle speed was eight times faster than her average swimming speed. Shannon took nine hours to complete the race. Calculate her average cycling speed.

Homework

Unit 5- Polynomial Functions

Lesson 17- Properties of Polynomials

A polynomial is a function that can be written like this:

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}...+a_2x^2+a_1x+a_0$$

Where $a_0, a_1, a_2, \dots, a_n$ are real numbers, $a_n \neq 0, n \in W$

- The values $a_0, a_1, a_2, \dots, a_n$ are called **coefficients** because they are the number in front of the **variable**, which is the letter x within the equation.
    - The coefficients can be positive, negative, zero, whole numbers, decimals or fractions.
- A **term** of the polynomial is any one piece of the sum, that is any a_ix^i.
- The coefficient of the highest power of x, which is a_n, is called the **leading coefficient.**
- The term independent of x, which is a_0, is the **constant term.**
- The value of n is the **degree** of the polynomial, which is the largest exponent.
- The **vertex** is the maximum or minimum point on quadratic functions.
- The **local max/ local min** are the top of the curve and bottom of the curve on a cubic function. It is a "local" max/min because a cubic function does not have an absolute max or min value.
- A **turning point** is the point where a graph changes direction. This would be the vertex on a quadratic function an the local max or min on a cubic function.

Examples of polynomial functions are:

- $f(x)=2x-1$ called a **linear** polynomial
- $f(x)=x^2+x-6$ called a **quadratic** polynomial
- $y=x^3+2x^2-5x-6$ called a **cubic** polynomial

Polynomials can be named by the highest exponent (MORE COMMON):

Highest exponent	Name
0	constant
1	linear
2	quadratic
3	cubic

Polynomials can be named by the number of terms:

Number of Terms	Name
1	monomial
2	binomial
3	trinomial

EX: For the following functions state the degree, leading coefficient and constant term.

a) x^3+2x^2-5x-6

b) $7+2x^3-5x^4-4x^2$

What does NOT qualify as a polynomial function:

- Roots of variables $\left(\sqrt{5x^3}\right)$
- Absolute values of variables $(|x|)$
- Negative or fractional exponents $(x^{-2})\ or\ \left(x^{\frac{3}{2}}\right)$
- Variables in the denominator $\left(\frac{4}{x^2}\right)$
- Any non-real coefficients $\left(\sqrt{-4}x^3\right)$

EX: Are these polynomials? If not, circle the part that makes it not a polynomial.

a) $f(x)=-4x^3+x^{\frac{3}{2}}-10$

b) $h(x)=3x^4+5x^{-2}+1$

c) $P(x)=x^4+5146x^3+\sqrt{6}x^2+148$

d) $P(x)=\sqrt{3x^5}+x^3-2x^2+8$

e) $P(x)=\sqrt{3x^4}+x^3-2x^2+8$

f) $P(x)=\dfrac{x+5}{x-2}$

g) $h(x)=3x^4+\sqrt{-5}x^2+1$

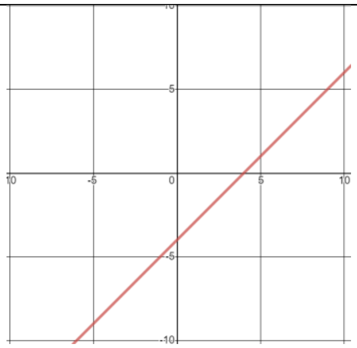
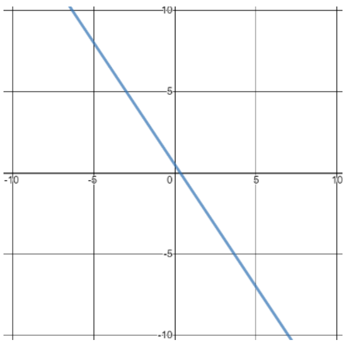
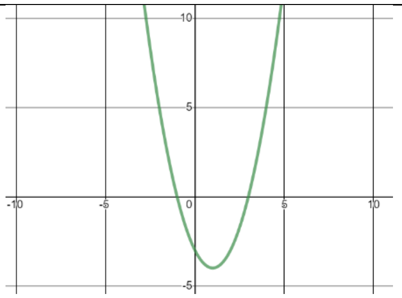
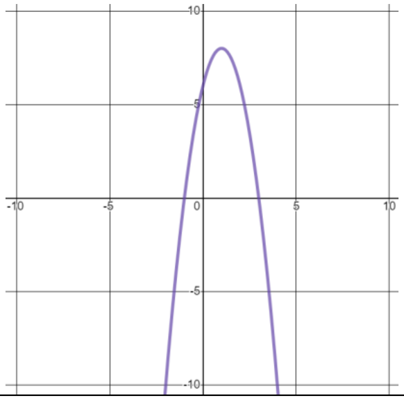
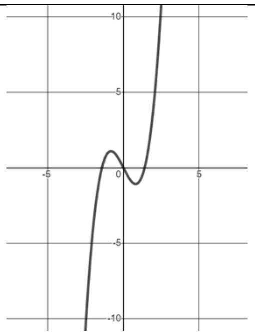
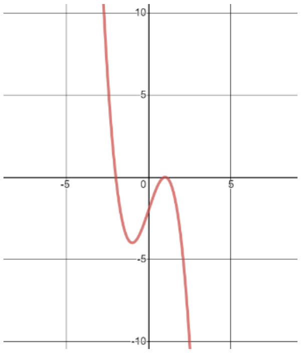
h) $h(x)=|x|+1$

Characteristics of Polynomial Functions

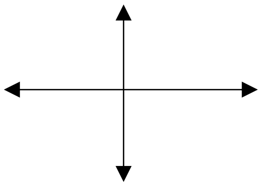
Polynomial functions and their graphs can be analysed by identifying the degree, end behaviour, domain and range, and the number of *x*-intercepts.

In Math 30-2 we focus on 3 types of polynomials:

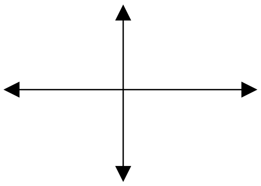
1. Linear: $f(x) = ax + b$
2. Quadratic: $f(x) = ax^2 + bx + c$
3. Cubic: $f(x) = ax^3 + bx^2 + cx + d$

Type of Function	Linear Function	Quadratic Function	Cubic Function
Polynomial	A function that contains only the operations of multiplication and addition, with real-number coefficients, whole number exponents, and 2 variables. The degree of the function is the greatest exponent of the function.		
General Form	$f(x) = ax + b$	$f(x) = ax^2 + bx + c$	$f(x) = ax^3 + bx^2 + cx + d$
Examples	$y = x - 4$ $f(x) = -\frac{3}{2}x + \frac{1}{2}$	$y = x^2 - 2x - 3$ $f(x) = -2x^2 + 4x + 6$	$y = x^3 - 2x$ $f(x) = -x^3 + 3x - 2$
Graphs (Notice the behavior of graphs if "a" is positive or negative)	 	 	 
Graph is called	Line	Parabola	Squiggle
Degree	1	2	3
Max # of x-intercepts	1	2	3
Turning points	None	1	2
y-intercept	Is the constant term (number at the end)	Is the constant term (number at the end)	Is the constant term (number at the end)
Domain	$x \in R$	$x \in R$	$x \in R$
Range	$y \in R$	$y \geq min$ Or $y \leq max$	$y \in R$
End Behaviour	-QIII to QI If "a" is Positive -QII to QIV If "a" is Negative	-QII to QI If "a" is Positive -QIII to QIV If "a" is Negative	-QIII to QI If "a" is Positive -QII to QIV If "a" is Negative

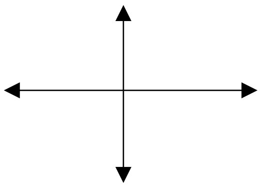
Positive “even” graphs look like this



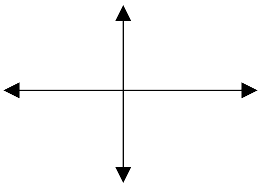
Positive “odd” graphs look like this



Negative "even" graphs look like this



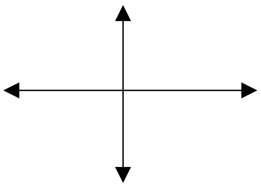
Negative "odd" graphs look like this



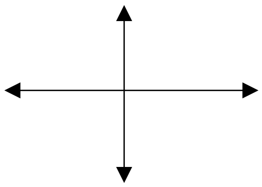
Every time the graph crosses the x-axis count as a zero. **The number of zeros gives the degree of the polynomial.**

There are some different cases of zeros that will be discussed in detail later: but for now, you need to know that these shapes could mean different numbers of zeros.

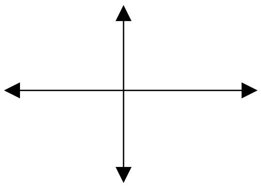
straight through



tangent



point of inflection



Factored-Form of Quadratic Functions

- If we factor the Quadratic Equation we get the following:

$$f(x) = 2x^2 + 4x - 16$$
$$f(x) = 2(x + 4)(x - 2)$$

  - From the factored form we can see that the x-intercepts are at

    $\boldsymbol{x = -4}$ **and** $\boldsymbol{x = 2}$
  - This can also be written as the points $(-4,0)$ and $(2,0)$

The Vertex of Quadratic Functions

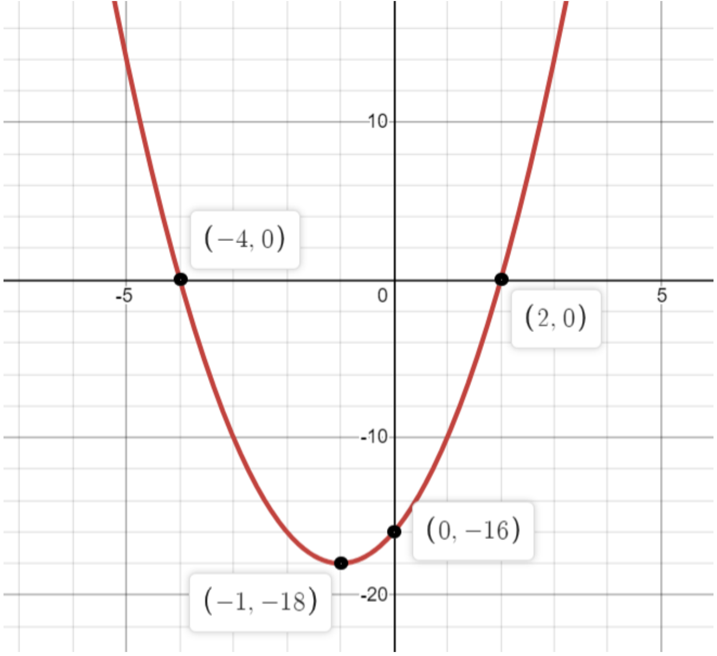
- We can find the vertex of the function $f(x) = 2(x+4)(x-2)$ two different ways
  - Graph the function and find the max or the min point
  - Find the vertex using algebra
    - Determine the x intercepts which are $(-4,0)$ and $(2,0)$
    - The x value of the vertex must be in the middle of the x-intercepts. Find the average between the x values. $\frac{(-4+2)}{2} = -1$
    - We now know the x value of our vertex, but we still need to find our y value.

$$V = (-1, y)$$

    - Substitute (-1) into the function $f(x) = 2x^2 + 4x - 16$

$$f(-1) = 2(-1)^2 + 4(-1) - 16$$
$$f(-1) = -18$$

    - **The vertex is at $V = (\mathbf{-1, -18})$**



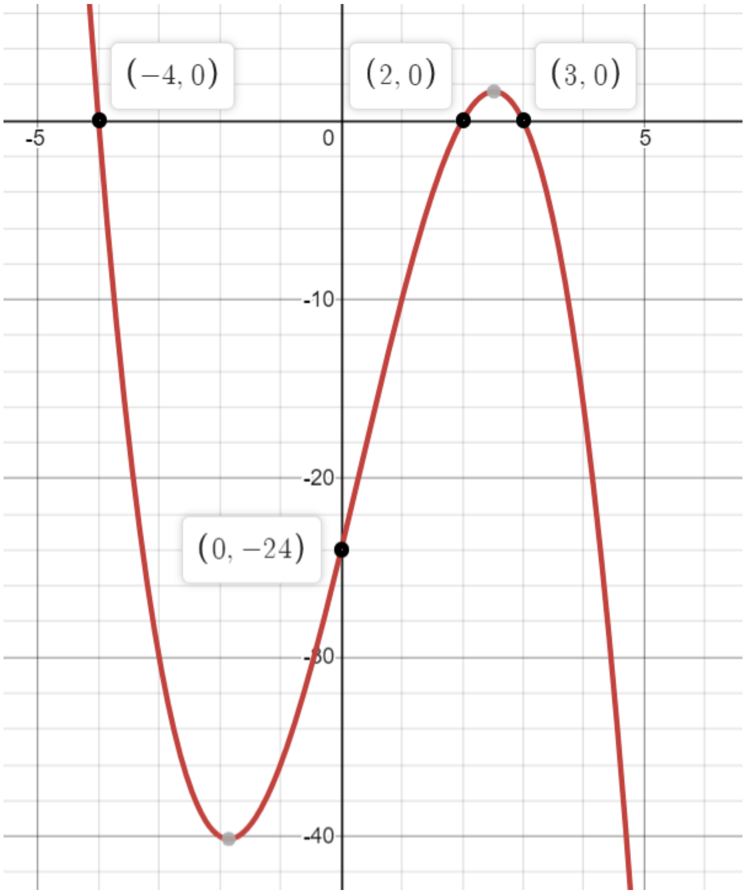
Factored-Form of Cubic Functions

$$f(x) = a(x+b)(x-c)(x-d)$$

You will not be required to factor cubic functions in Math 30-2. However, if you are given a factored cubic function like the one below, you need to describe the following properties

$$f(x) = -(x+4)(x-3)(x-2)$$

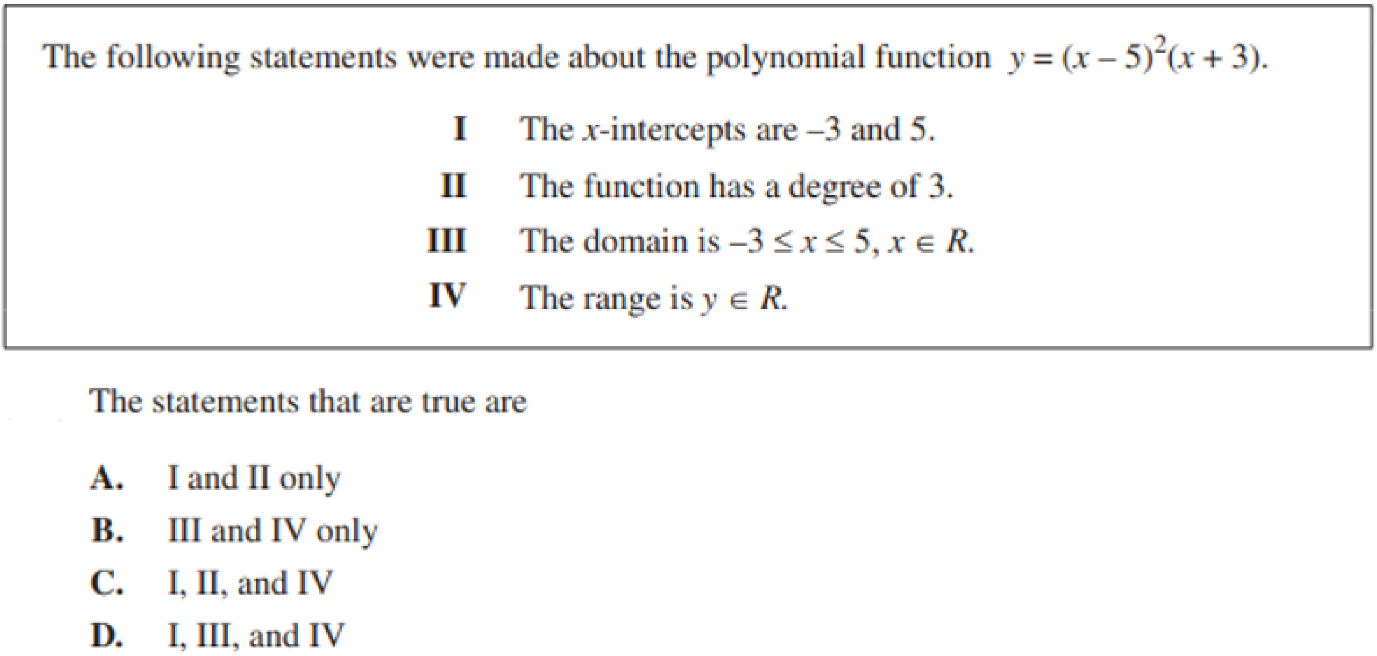
- **x-intercepts**
  The x-intercepts from the function above are $x = -4, 3, 2$. These can also be written as the points: $(-4,0), (3,0), (2,0)$. Notice that the "signs" have been **swapped**.
- **y-intercept**
  Substitute (0) in for x.
  $$f(0) = -(0+4)(0-3)(0-2)$$
  $$f(0) = -(4)(-3)(-2)$$
  $$\boldsymbol{f(0) = -24}$$
- **Domain**
  Unless the context of the question suggests that we make a restriction,
  $$\boldsymbol{Domain: x \in R}$$
- **Range**
  Unless the context of the question suggests that we make a restriction,
  $$\boldsymbol{Range: x \in R}$$
- **Sketch the graph**
  The "a" value is negative so the graph starts in QII and ends in QIV



EX: Determine the x-intercepts and y-intercept for the following function: $y=(x-1)(x+2)(x+3)$

EX: Describe the graph of $f(x)=-(x+1)^2(x-2)$. Include the intercepts and turning points in your description.

EX:



The following statements were made about the polynomial function $y=(x-5)^2(x+3)$.

I The x-intercepts are -3 and 5.

II The function has a degree of 3.

III The domain is $-3 \le x \le 5,\ x \in R$.

IV The range is $y \in R$.

The statements that are true are

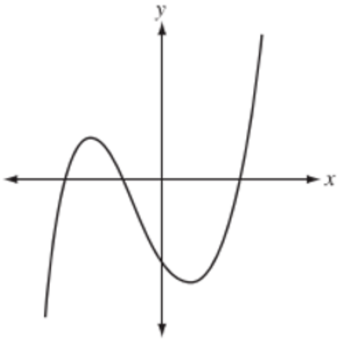
A. I and II only
B. III and IV only
C. I, II, and IV
D. I, III, and IV

EX:

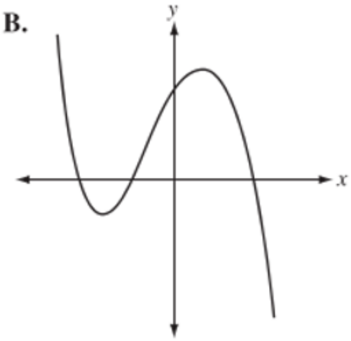
An equation of a particular cubic function is $y = (x - m)(x - n)(x + p)$, where m, n, and p are different whole numbers.

Which of the following could be the graph of this polynomial function?

A.



B.



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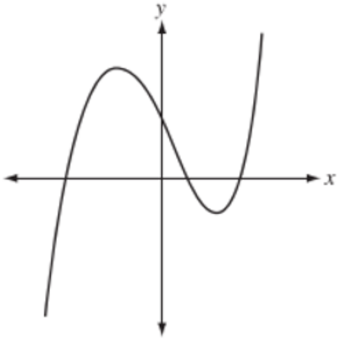
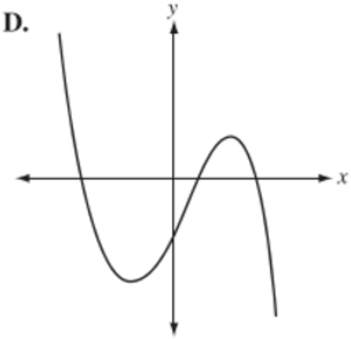


D.



Homework

Lesson 18- Polynomial Applications

When we are given a real-world problem, specific parts of the graph describe certain events in the question.

STEPS FOR FINDING YOUR X-VALUE WITH 2 Y-VALUES:

1. Insert the one equation into Y1, the other equation into Y2.
2. Graph the functions
3. Calculate the **intersection point** for the two graphs
4. The x-values of the intersection points are your answers! We don't care about the y-values.

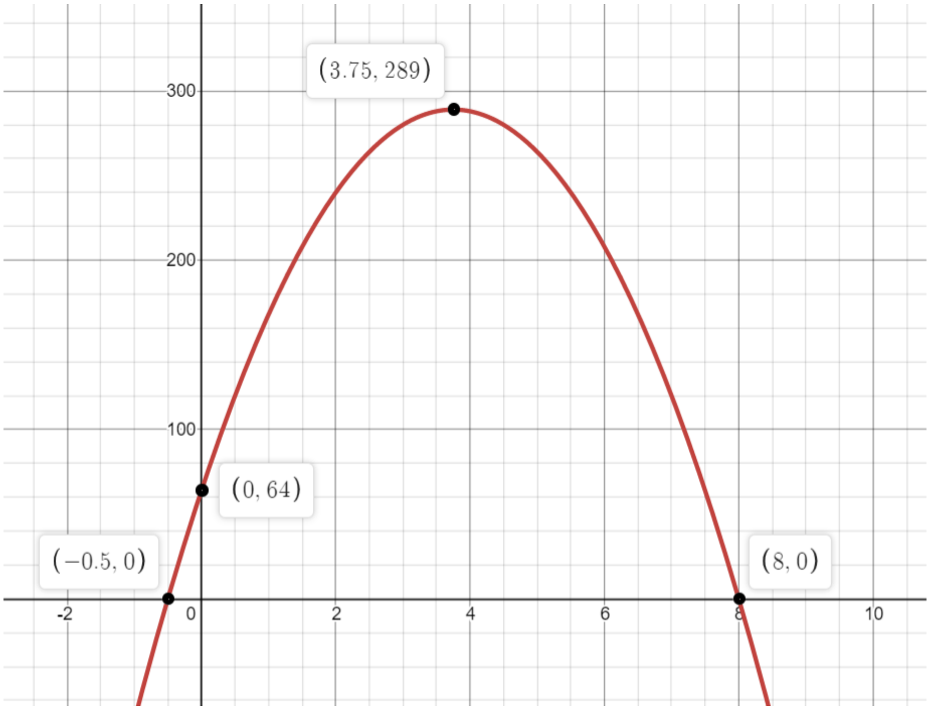
STEPS FOR FINDING YOUR X-INTERCEPTS:

1. Insert the whole equation into Y1.
2. Graph the function
3. Calculate the **zeros** for the graph
4. The x-values of the zeros are your answers! We don't care about the y-values.

STEPS FOR FINDING YOUR Y-VALUE AT A SPECIFIC X-VALUE:

1. Insert the whole equation into Y1.
2. Graph the function
3. Calculate the **value** for the graph by subbing in your x value in the flashing x=___
4. The y-value of the point is your answer! We don't care about the x-value.

EX: A person standing close to the edge on top of a 64-foot building throws a ball vertically upward. The quadratic function $h(t) = -16t^2 + 120t + 64$ models the ball's height about the ground, h, in feet, t seconds after it was thrown.



Sketch the graph (You can your calculator or find the properties below to make a rough sketch.

a) How high is the football after 4 seconds?

You can use the "trace" function on your calculator to find y when x is 4.

Or

Substitute 4 in for x and solve the function. $f(4) = 16(4)^2 + 120x + 64$

$$\boldsymbol{f(4) = 288}$$

b) What is the max height of the ball?

The max height is the **y value** of the vertex. Graph the function to find the vertex.

The max height is 289 feet.

c) At what time did the ball reach its max height

The **x value** of the vertex is the time at the max height.

The time at the max height is 3.75 seconds

d) What part of the graph relates to the height of the building?

The height of the building is the y value at the start (time = 0s), which is the **y-intercept**.

The height of the building is 64 feet

e) How long was the ball in the air?

The ball was in the air at zero seconds and ended when the ball hit the ground (the second x-intercept).

The ball was in the air for 8 seconds

f) What is the practical domain of the question?

The ball was thrown at time zero, so for our function, time must be greater than or equal to zero. When the ball hits the ground, our function wouldn't make sense anymore so time should be less than 8 seconds.

Time can start at 0 but needs to be less than 8.

$$\boldsymbol{Domain: 0 \leq t < 8\,, t \in R.}$$

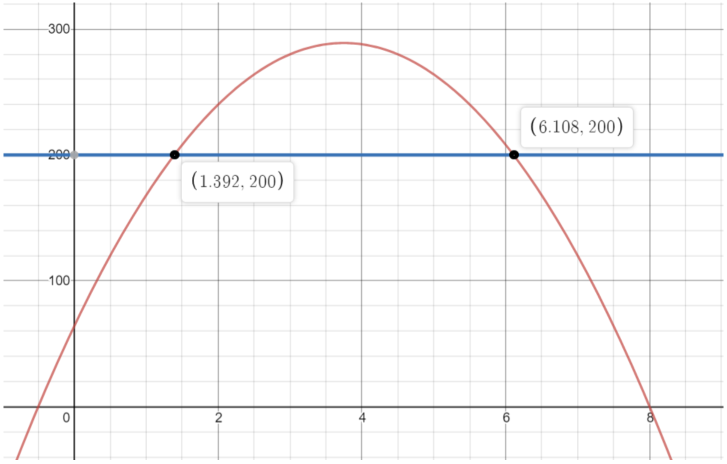
g) What is the practical range of the question?

The ball never goes above 289 feet, so the height is less than or equal to 289.

The ball can't fall below the ground so height is greater than 0 feet.

$$\boldsymbol{Range: 0 < h \leq 289}$$

h) At what time(s) does the ball reach a height of 200 feet?



Graph a second line at h=200. This is the horizontal line. The x values of where the two functions meet is the time(s) the function is at a height of 200 feet.
The two times that the ball is at 200 feet are $t = 1.392s\ and\ t = 6.108s$

EX

Water is being pumped into a 15-gallon tank. Once the volume of water in the tank reaches a certain amount, the tank begins to drain and continues draining until the water is completely gone. The volume of water in the tank can be modelled by the function

$$y = -2t^2 + 5t + 7$$

where y represents the volume of water in the tank in gallons and t represents the time in hours after noon on a particular day.

To determine the volume of water in the tank at noon, the characteristic of the graph of the function that should be analyzed is the

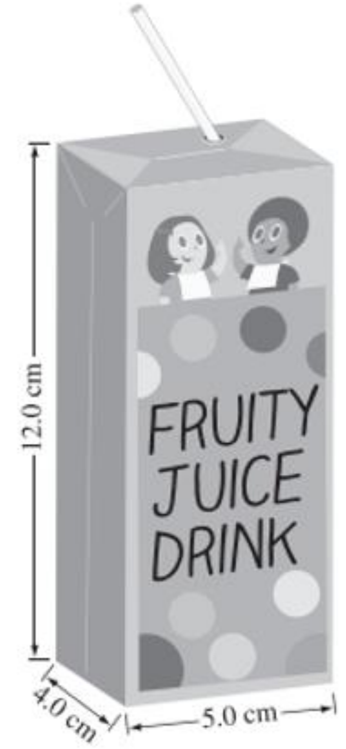
y-intercept
positive t-intercept
t-coordinate of the vertex
y-coordinate of the vertex

Which of the following rows describes the **most appropriate** domain and range of the graph of the function in this context?

Row	Domain	Range
[redacted]	$t \in R$	$y \leq 10.125$
[redacted]	$t \in R$	$0 \leq y \leq 10.125$
[redacted]	$0 \leq t \leq 3.5$	$y \leq 10.125$
[redacted]	$0 \leq t \leq 3.5$	$0 \leq y \leq 10.125$

EX:

A company wants to change the dimensions of a particular box, which will increase the volume. The box is currently 5.0 cm × 4.0 cm × 12.0 cm, as shown below.



To create the new box, the company will increase each dimension by the same amount. The volume of a box can be calculated by using the formula $V = l \cdot w \cdot h$.

The company does not want the volume of the box to exceed 1 000 cm^3. Determine the largest amount by which each dimension of the box can be increased, to the nearest tenth of a centimetre.

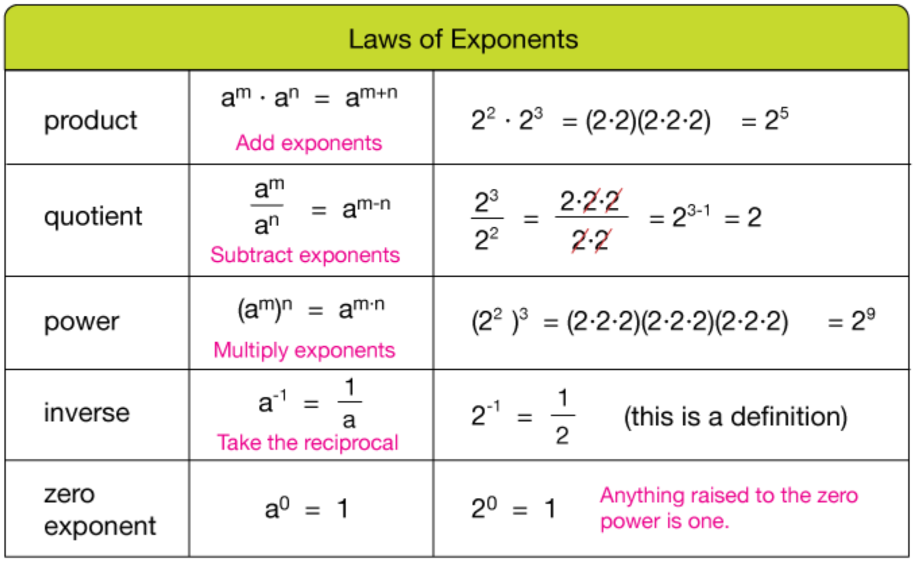
Homework

Unit 6- Exponential and Logarithmic Functions

Lesson 19- Graphs of Exponential and Logarithmic Functions

Review

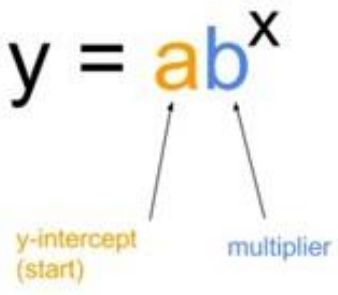
- Exponent Law:



Laws of Exponents		
product	$a^m \cdot a^n = a^{m+n}$ Add exponents	$2^2 \cdot 2^3 = (2\cdot2)(2\cdot2\cdot2) = 2^5$
quotient	$\frac{a^m}{a^n} = a^{m-n}$ Subtract exponents	$\frac{2^3}{2^2} = \frac{2\cdot\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}} = 2^{3-1} = 2$
power	$(a^m)^n = a^{m\cdot n}$ Multiply exponents	$(2^2)^3 = (2\cdot2\cdot2)(2\cdot2\cdot2)(2\cdot2\cdot2) = 2^9$
inverse	$a^{-1} = \frac{1}{a}$ Take the reciprocal	$2^{-1} = \frac{1}{2}$ (this is a definition)
zero exponent	$a^0 = 1$	$2^0 = 1$ Anything raised to the zero power is one.

  - *These laws are a review from previous math course. They are NOT provided on your formula sheet, but you are required to know them.*

- Exponential Function:
  - **a**: represents the **initial amount/ starting amount** of the question. Also represents your **y-intercept**.
  - **b**: represents the **base/multiplier**, which is what the equation grows or decays by.



Investigating Exponential Behaviours:

Let's begin by looking at the function $f(x) = 2^x$.

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice the following:

- The function (y-value) is positive for all values of x.
- As x increases, the function grows faster and faster (the rate of change INCREASES).
- As x decreases, the function grows smaller, approaching zero.
- This is an example of **exponential growth.**

Now let's look at the function $g(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
g(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

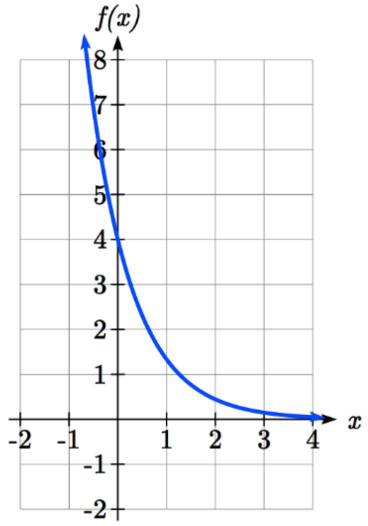
Notice the following:

- The function is positive for all values of x.
- As x increases, the function grows smaller, approaching zero (the rate of change DECREASES).
- As x decreases, the function grows faster.
- This is an example of **exponential decay.**

Characteristics of Exponential Functions:

- Characteristics refer to concepts such as domain, range, x-intercepts, y-intercepts, end behaviour, and horizontal and vertical asymptotes.
- The following are a list of characteristics for exponential functions that you are require to know given the function $y = ab^x$
    - The y-intercept is the a value within the equation
    - There are NO x-intercepts
    - The x-axis forms a horizontal asymptote (y=0)
    - The domain is $x \in \mathbb{R}$
    - The range is $\{y > 0, y \in \mathbb{R}\}$ if $a > 0$ and $\{y < 0, y \in \mathbb{R}\}$ if $a < 0$
    - For $a > 0$:
        - The value of b affects the steepness of the graph as x increases
            - When $b > 1$, the curve rises sharply as b increases, representing a growth
            - When $0 < b < 1$, the curve falls sharply as b decreases, representing a decay
- When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$

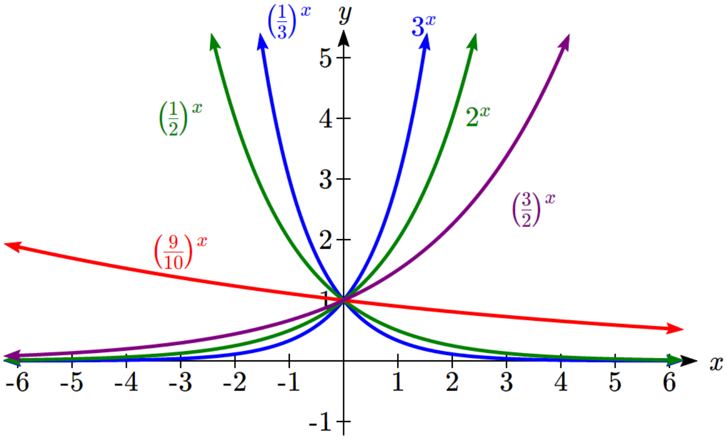
EX: Sketch the graph of $f(x) = 4\left(\frac{1}{3}\right)^x$



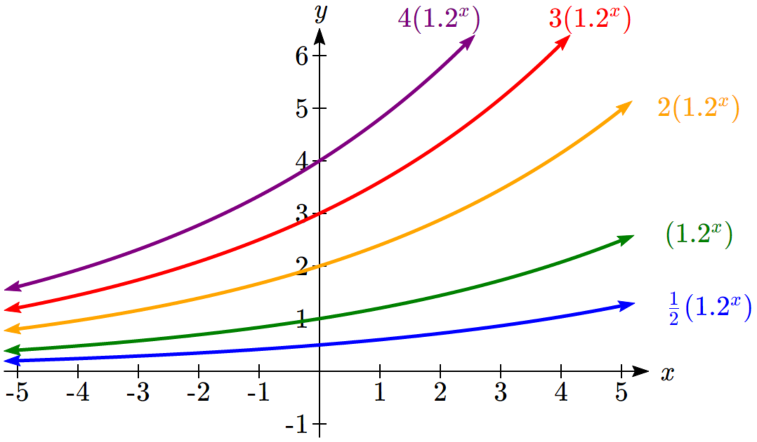
- This graph will have a vertical intercept at $(0,4)$, and pass through the point $\left(1, \frac{4}{3}\right)$
- Since $b < 1$, the graph will be a decrease towards zero.

To get a better feeling for the effects of a and b on the graph, examine the sets of graphs below.

- The first set shows various graphs, where a remains the same and we only change the value of b.
    - Notice, the closer the value of b is to 1, the less steep the graph will be.



- In the next set of graphs, a is altered and our value for b remains the same.
    - Notice that changing the value for a changes the y-intercept.



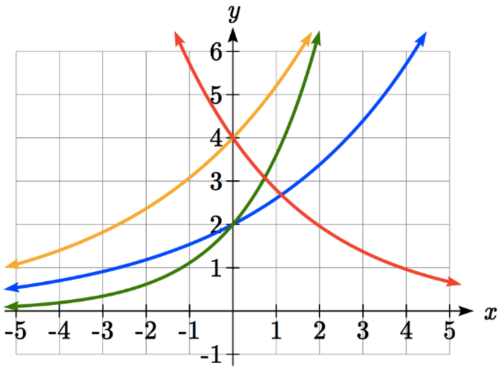
EX: Match each equation with its graph:

$f(x) = 2(1.3)^x$

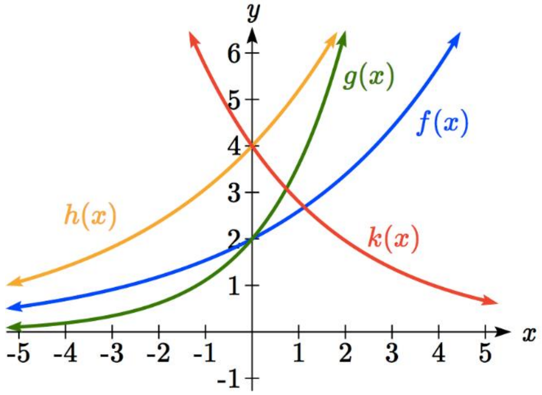
$g(x) = 2(1.8)^x$

$h(x) = 4(1.3)^x$

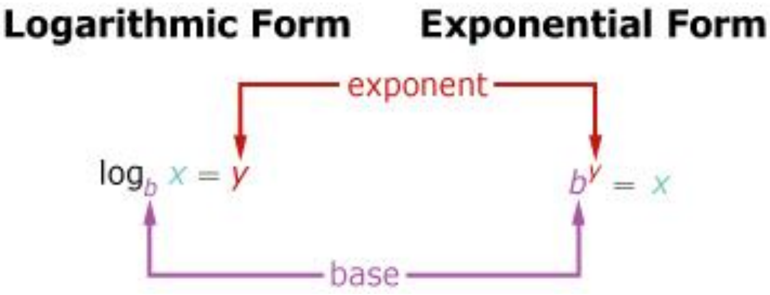
$k(x) = 4(0.7)^x$



- The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph.
- The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at $(0,4)$.
- The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at $(0,2)$, but since $g(x)$ has a larger growth factor, we can identify it as the graph increasing faster.



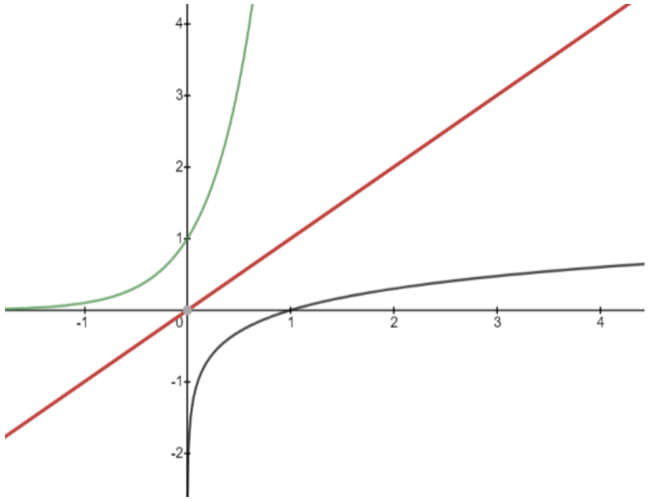
- Logarithmic and Exponential Function:
    - **b**: represents the **base/multiplier** in both exponential and logarithmic form. It is just written in a different spot and different size.
    - **y**: *Logs solve for exponents.* This means that the exponent in the exponential form is now what the equation EQUALS in the logarithmic form.
    - **x**: whatever the exponent equals, is now the larger number by the logarithmic function. In logarithmic form the "x" is known as the argument.



Investigating Logarithmic Functions:

A logarithmic function is the inverse of an exponential function.

- An inverse function has x values that were the y values from the original and y values that were x values from the original.
- The inverse of $y = 10^x$ is $x = 10^y$.
- You can also write the second expression using a logarithm $y = log_{10}x$.



The graph of a reciprocal function will always be a reflection about the line $y = x$.

The point $(1, 10)$ on the graph of $y = 10^x$ would be the point $(10, 1)$ on the graph of $y = log_{10}x$.

Recall the exponential function $f(x) = 2^x$ produces this table of values:

x	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x) = log_2(x)$ produces this table of values:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

Notice the following:

- As the input (x-value) increase, the output (y-value) increases
- As the input increases, the output increases more slowly
- Since the exponential function only outputs positive values, the logarithmic can only accept positive values as inputs.
- Since the exponential function can accept all real numbers as inputs, the logarithm can output any real numbers.

Characteristics of Logarithmic Functions:

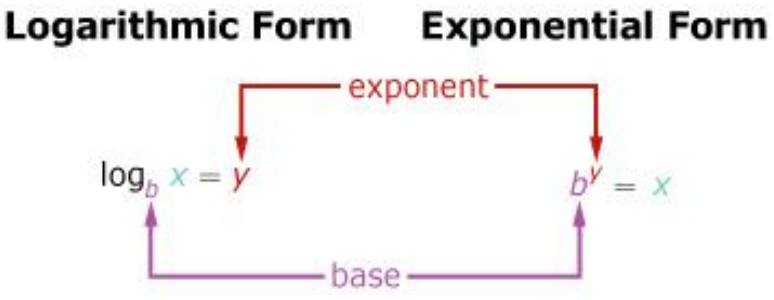
- Characteristics refer to concepts such as domain, range, x-intercepts, y-intercepts, end behaviour, and horizontal and vertical asymptotes.
- The following are a list of characteristics for exponential functions that you are require to know given the function $log_b x = y$
    - Logarithmic form is the inverse of exponential form.
        - For example: The inverse of the follow exponential equation is: $y = 5^x \rightarrow x = 5^y$, but we do not typically write y as an exponent. Therefore, we write it in the form $y = log_5 x$
    - There is NO y-intercepts
    - Typically, the x-intercept is at (1,0)
    - The y-axis forms a vertical asymptote (x=0)
    - The domain is $\{x > 0, x \in \mathbb{R}\}$
    - The range is $\{y \in \mathbb{R}\}$
    - Restrictions: $b > 0,\ b \neq 1$
- When sketching a general logarithmic with base b, it can be helpful to remember that the graph will pass through the points $(1,0)$ and $(b, 1)$.

Homework

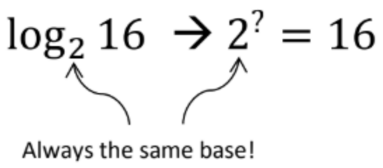
Lesson 20- Evaluating Logarithms

Converting between Exponents and Logarithms:

- $y = b^x \ SAME \ AS \ x = log_b y$


Logarithmic Form
Exponential Form
exponent
$\log_b x = y$
$b^y = x$
base

EX:


$\log_2 16 \rightarrow 2^? = 16$
Always the same base!

- When we write $log_2 16$, we are really asking “2 to the power of what is 16?”
- We can do this without a calculator.
    - 2 to the power of **4** is 16, therefore, $log_2 16 = 4$

Logs solve for exponents!

EX: Write the following exponential equations as logarithmic equations:

1. $2^3 = 8$
    a. The exponent of 3 is what the log EQUALS.
    b. The base of 2 is the same for both exponents and logs.

$$\boldsymbol{log_2 8 = 3}$$

2. $5^2 = 25$

$$\boldsymbol{log_5 25 = 2}$$

3. $10^{-4} = \frac{1}{10000}$

$$\boldsymbol{log_{10} \frac{1}{10000} = -4}$$

EX: Write the following logarithmic equations as exponential equations:

1. $log_6 \sqrt{6} = \frac{1}{2}$
    a. The $\frac{1}{2}$ is what the log equals, therefore it is the exponent.
    b. The base of 6 is the same for both exponents and logs.

$$\boldsymbol{6^{1/2} = \sqrt{6}}$$

2. $log_3 9 = 2$

$$\boldsymbol{3^2 = 9}$$

EX: Convert the following from exponential form to logarithmic form:

a) $3^x = 81$

b) $36^{\frac{1}{2}} = x$

c) $x^{-3} = 343$

EX: Convert the following from logarithmic form to exponential form:

a) $log_2 32 = 5$

b) $log\ 0.0001 = -4$

c) $log_{\frac{1}{2}} 2x = -3$

This conversion between exponents to logs and logs to exponents allows us to solve different equations.

- If we are given a variable in the exponent, convert it to logarithmic form and solve from there.
- If we are given a variable in the logarithms argument, convert it to exponential form and solve from there.

EX: Solve $log_4(x) = 2$ for x.

- By rewriting this expression as an exponential equation, the x will be isolated.

$$4^2 = x$$
$$\mathbf{16 = x}$$

EX: Solve $2^x = 10$ for x.

- By rewriting this expression as a logarithm equation, the x will be isolated.

$$x = log_2 10$$

- To solve this in your calculator, you will utilize the "Change of Base" formula:

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$

Log of argument OVER Log of base

$$\boldsymbol{x \approx 3.32}$$

EX: Evaluate $4log_a a$

- If we look closely at $log_a a$, and thich of this in terms of an exponent, we get

$$a^? = a$$

- This simply means "a to the power of what equals a"

$$a^1 = a$$

- So $log_a a = 1$
    - ***This is applicable for any log that has the same value in the argument and the base.***
- To solve we would rewrite $log_a a$ as 1

$$4(1)$$
$$\mathbf{4}$$

Homework

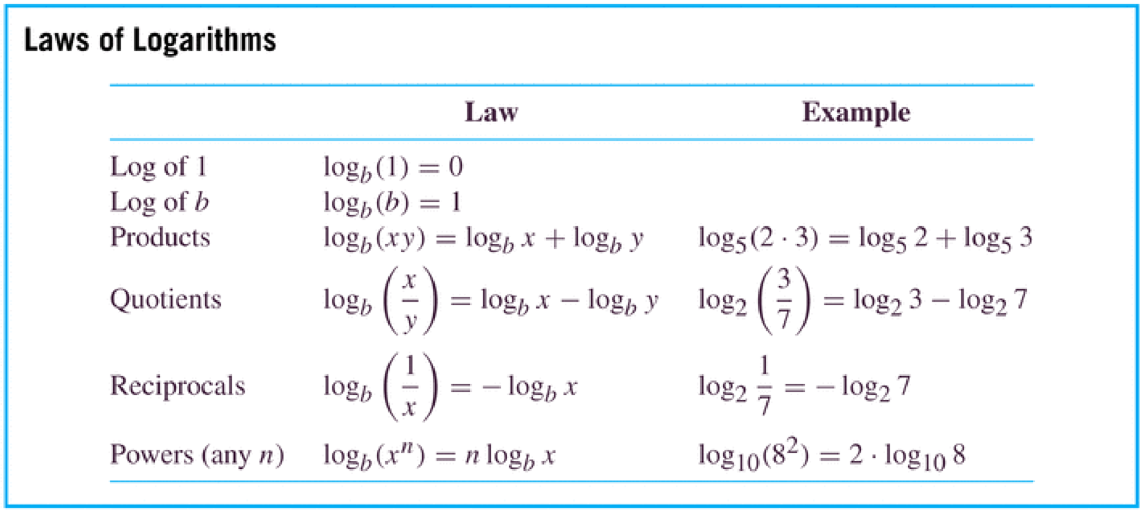
Lesson 21- Laws of Logarithms

Common and Natural Logarithms:

- The **common log** is the logarithm with the base of 10, as is typically written as $\log(x)$
  - Notice how the 10 is "invisible"
- The **natural log** is the logarithm with the base of e, and is typically written as $\ln(x)$

	Logarithm	=	**Exponent**
	$\log_a N = x$	$\longleftrightarrow$	$N = a^x$
(Common Log)	$\log N = x$	$\longleftrightarrow$	$N = 10^x$
(Natural Log)	$\ln N = x$	$\longleftrightarrow$	$N = e^x$

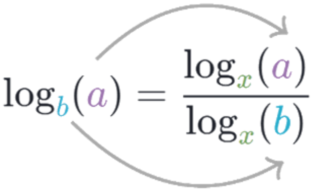
Logarithmic functions are the inverse of exponential function; therefore, we can invert the laws for exponents and create the Laws of Logarithms.



Laws of Logarithms

	Law	Example
Log of 1	$\log_b(1) = 0$	
Log of b	$\log_b(b) = 1$	
Products	$\log_b(xy) = \log_b x + \log_b y$	$\log_5(2 \cdot 3) = \log_5 2 + \log_5 3$
Quotients	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log_2\left(\frac{3}{7}\right) = \log_2 3 - \log_2 7$
Reciprocals	$\log_b\left(\frac{1}{x}\right) = -\log_b x$	$\log_2 \frac{1}{7} = -\log_2 7$
Powers (any n)	$\log_b(x^n) = n \log_b x$	$\log_{10}(8^2) = 2 \cdot \log_{10} 8$

Change of Base:



$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$

**Note: these formulas ARE on your formula sheet.*

You are required to know how to **expand, simplify or algebraically evaluate** a logarithmic expression.

EX: Rewrite $log_3(25)$ using the exponent (power) property for logs.

- Since $25 = 5^2$
- We can then replace the 25 with 5^2

$$log_3(5^2)$$

- Now bring the exponent forward

$$\boldsymbol{2log_3(5)}$$

EX: Evaluate $log_a(a^5)$

- Start with bringing the exponent forward

$$5log_a a$$

- Utilize $log_a a = 1$ from last lesson

$$5(1)$$
$$\mathbf{5}$$

EX: Write the following expressions as single logarithms where $b > 1$.

1. $log_b(2x) + log_b(3x)$
    a. Since the bases are the same, multiply the arguments and mold it into one log.

$$log_b(2x)(3x)$$
$$\boldsymbol{log_b(6x^2)}$$

2. $2log_b x - log_b y$
    a. You always need to get the logs into logs WITH exponents.

$$log_b x^2 - log_b y$$

    b. Now divide the arguments and mold it into one log.

$$\boldsymbol{log_b \frac{x^2}{y}}$$

EX: Rewrite each expression as a simplified single logarithmic expression. Ensure your exponents are written in positive form.

a) $log_2 x^3 - 4log_2 x - log_2\sqrt{x}$

b) $4log_6 y^2 + log_6 y - \frac{2}{3}log_6 y$

c) $log(x-3) + log(x+4)$

EX: Evaluate $2\log(5) + \log(4)$ without a calculator by first rewriting as a single logarithm.

- Bring the exponent back up on the first log

$$\log(5^2) + \log(4)$$

- Since 5^2 simplifies to 25, rewrite it as such

$$\log(25) + \log(4)$$

- We now utilize the sum of logs property to mold them into one log

$$\log(25 \cdot 4) = \log(100)$$

- Using the property of "Logs solve for exponents", let's evaluate this

$$log_{10}100 \rightarrow 10^{?} = 100$$
$$10^2 = 100$$
$$\boldsymbol{Therefore, our\ answer\ is\ 2}$$

EX: Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs

- First, noticing we have a quotient, we will utilize the difference property of logs to write

$$\ln(x^4y) - \ln(7)$$

- Now we will address the product in the first term by using the sum property of logs

$$\ln(x^4) + \ln(y) - \ln(7)$$

- Finally, if there are exponents, move them in front of the term

$$\boldsymbol{4\ln(x) + \ln(y) - \ln(7)}$$

EX: Expand the following:

a) $log_4 \frac{x^3y}{4z}$

b) $log_7\sqrt{x^2y^3}$

c) $log\,\frac{1000\sqrt[4]{x^5}}{z^3}$

EX: Use the laws of logarithms to determine the value of each of the following expressions

1. $log_6 3 + log_6 12$

$$log_6 36$$
$$\mathbf{2}$$

2. $log520 - log52$

$$log10$$
$$\mathbf{1}$$

EX: First simplify the logarithmic expression into a single logarithmic, then evaluate for the given value.

a) $log_x 10 + log_x 75 - (log_x 2 + log_x 3)$ if $x = 5$

b) $log_z 1 + \frac{1}{2} log_z 100 - log_z 0.00001^2$ if $z = 10$

Homework

Lesson 22- Solving Exponential Equations without Logarithms

Solving an equation means to determine the value of the variable (typically x) that makes the equation true. The problem with exponential functions is that the variable is an exponent. There are three methods that can be used to solve exponential equations:

- **Rewriting Bases** – used if the bases are the same, or are powers of one another (when solving algebraically)
- **Graphically** – used if the bases are not the same, or are not powers of one another (when solving multiple choice or numeric response answers)
- **Using Logarithmic Equations** – used if the bases are the same or not the same (when solving algebraically) *We will do this in the next lesson.*

Changing Bases:

EX: Covert each of the following to the base indicated

a) 9^{2x} to base 3

b) 125^{2-x} to base 5

c) $8 \cdot 16^{x}$ to base 2

d) $\frac{1}{512^{3x}}$ to base 2

e) $\left(\frac{16}{81}\right)^{x+5}$ to base $\frac{2}{3}$

STEP FOR SOLVING:

1. Write each side of the equation in the same base
2. If necessary, use exponent laws so that each side of the equation contains **only one base**
3. Equate the exponents on each side of the equation
4. Determine the value of the variable

EX: Solve $8^x = 128$

- We can express 8 and 128 as powers of 2.

$$2^{3x} = 2^7$$

- Now that the bases are equal, equate the exponents and solve.

$$3x = 7$$
$$\boldsymbol{x = \frac{7}{3}}$$

EX: Solve $4^{x-8} = 16$

- We can express 16 as a power of 4. Notice, you don't have to change both sides!

$$4^{x-8} = 4^2$$

- Now that the bases are equal, equate the exponents and solve.

$$x - 8 = 2$$
$$\boldsymbol{x = 10}$$

EX: Solve $10^{x-3} = 100^{3x+11}$

- We can express 100 as a power of 10.

$$10^{x-3} = 10^{2(3x+11)}$$

- Be sure to FOIL in the 2 on the second term, then solve!

$$x - 3 = 6x + 22$$
$$-25 = 5x$$
$$\boldsymbol{-5 = x}$$

EX: Solve the following for x.

a) $10^{x+4} = 1000^{x-4}$

b) $25^{2x} = 125^{x-1}$

c) $8^{x+2} = \left(\frac{1}{4}\right)^{x+3}$

d) $121^{2x-5} = 1$

e) $21\left(\frac{1}{9}\right)^{4x} = 7$

Graphing:

Two methods are use, one for when you have an equation on either side of the equal side, and one for when you only have one equation all on one side of the equation sign.

STEPS FOR 2 EQUATIONS:

1. Insert the left side of the equation into Y1, the right side of the equation into Y2.
2. Graph the functions
3. Calculate the **intersection point** for the two graphs
4. The x-values of the intersection points are your answers! We don't care about the y-values.

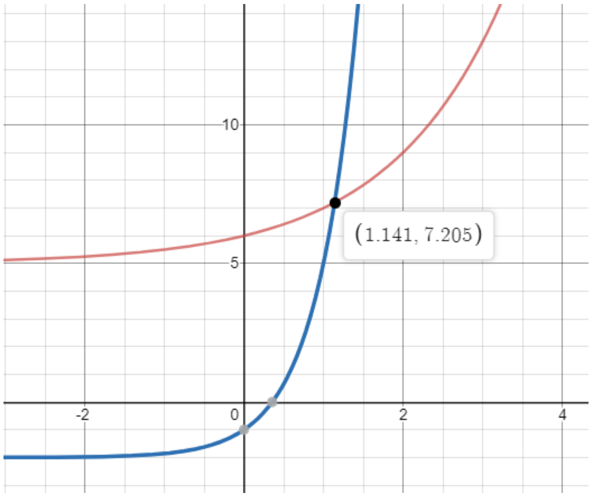
STEPS FOR 1 EQUATION:

1. Insert the whole equation into Y1 (once you have moved all the terms to one side).
2. Graph the function
3. Calculate the **zeros** for the graph
4. The x-values of the zeros are your answers! We don't care about the y-values.

EX: Solve $2^{(x+5)} = 7^{(x-2)}$

- Graph both sides of the equation in your calculator
    - ○ $y_1 = 2^{(x+5)}$
    - ○ $y_2 = 7^{(x-2)}$
- Find the intersection and take the x value

$$\boldsymbol{x \approx 1.141}$$



EX: Solve the following for x. Round your answers to the thousandths.

a) $5^{x} = 3^{-x+2}$

b) $4^{-2x} = \left(\frac{1}{2}\right)^{3-x}$

Homework

Lesson 23- Solving Exponential Equations Using Logarithms

Solving Exponential Equations:

REWRITING BASES:

STEPS FOR SOLVING:

1. Write each side of the equation in the same base
2. If necessary, use exponent laws so that each side of the equation contains **only one base**
3. Equate the exponents on each side of the equation
4. Determine the value of the variable

USING LOGARITHMIC EQUATIONS:

STEPS FOR SOLVING: (when there is ONLY an exponent on one side of the equation)

1. Rewrite the equation from exponential to logarithmic form.
2. You may now need to utilize different ways to solve:
    a. You may be able to simply solve already (using algebra)
    b. Utilize your graphing calculator

STEPS FOR SOLVING: (when there are exponents on BOTH sides of the equation)

1. Log both sides of the equation.
2. Use Log Laws to expand the equations.
3. Get everything with a variable to one side of the equation sign and everything without a variable to the other side of the equation sign.
4. Factor out the variable on the one side.
5. Divide over the equation that does not hold the variable any more, isolating the variable on the one side.

EX: Solve the following:

1. $9^{2x} = 3$
    a. Because they can easily be transformed into similar bases, I would utilize this application for this question.

$$3^{2(2x)} = 3^1$$
$$4x = 1$$
$$\boldsymbol{x = \frac{1}{4}}$$

2. $2^{(x-1)} = 4^{(3x-1)}$
    a. Because they can easily be transformed into similar bases, I would utilize this application for this question.

$$2^{(x-1)} = 2^{2(3x-1)}$$
$$x - 1 = 6x - 2$$
$$1 = 5x$$
$$\boldsymbol{\frac{1}{5} = x}$$

3. $10 = 3^x$
    a. You can do this 2 ways, the first will be converting this exponential equation into a logarithmic one.

$$x = log_3 10$$
$$x = \frac{log10}{log3}$$
$$\boldsymbol{x \approx 2.1}$$

b. The second will be by logging both sides of the equation.

$$log10 = log3^x$$
$$log10 = xlog3$$
$$\frac{log10}{log3} = x$$
$$\boldsymbol{x \approx 2.1}$$

4. $3 \cdot 2^{(x-1)} = 24$
   a. Be sure to always **isolate the base and it's exponent**.

$$\frac{\cancel{3} \cdot 2^{(x-1)}}{\cancel{3}} = \frac{24}{3}$$

$$2^{(x-1)} = 8$$

- At this stage, you can either convert to similar bases, convert into a log, or log both sides. All applications work, choose whichever one you are most comfortable with.

$$(x-1) = log_2 8$$
$$x = log_2 8 + 1$$
$$\boldsymbol{x = 4}$$

EX: Solve the following both as exact values and as a decimal rounded to the thousandths.

a) $2^x = 64$

d) $10^{x-2} = 2^{-x+4}$

b) $3^{2x} = 27^{x-4}$

e) $2(3^{x+4}) = 7^{\frac{x}{2}}$

c) $4^x = 5^{x-2}$

Homework

Lesson 24- Applications of Exponential and Logarithmic Functions

Exponential Growth vs. Decay

Any type of question that deals with exponential growth or decay can be modeled using the formula $y = ab^x$ where:

- a represents the initial amount/ starting point
- b represents the rate of change
  - If it is a growth question $b > 1$
  - If it is a decay question $0 < b < 1$
- x represents the unit time
- y represents the amount after x number of time

EX: If you saw 5 lilies on a pond and determined that the lilies reproduce by 35% per week, what is the general equation for the number of lilies after x number of weeks?

- First start with your a value- this is always what amount you start with

$$a = 5$$

- Second, find your b value- this is a growth question, therefore you start with 100% and **add** 35%. Be sure to switch the percent into a decimal before placing it into your equation.

$$b = 1.35$$

$$\boldsymbol{y = 5(1.35)^x}$$

EX: You see a pond full of 2000 lilies, but a disease has started to kill many of them. You estimate that 30% of the lilies are dying each week. What is the general equation for the number of lilies after x number of weeks?

- First start with your a value

$$a = 2000$$

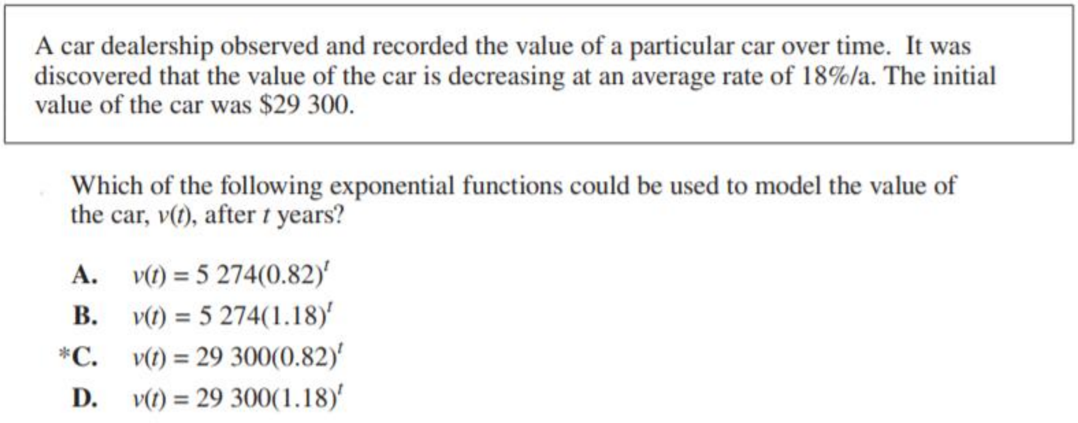
- Second, find your b value- this is a decay question, therefore you start with 100% and **subtract** 30%. Be sure to switch the percent into a decimal before placing it into your equation.

$$b = 0.70$$

  - If 30% of the lilies dye each week, then $\boldsymbol{70\%}$ **are still alive after each week.**

$$\boldsymbol{y = 2000(0.7)^x}$$

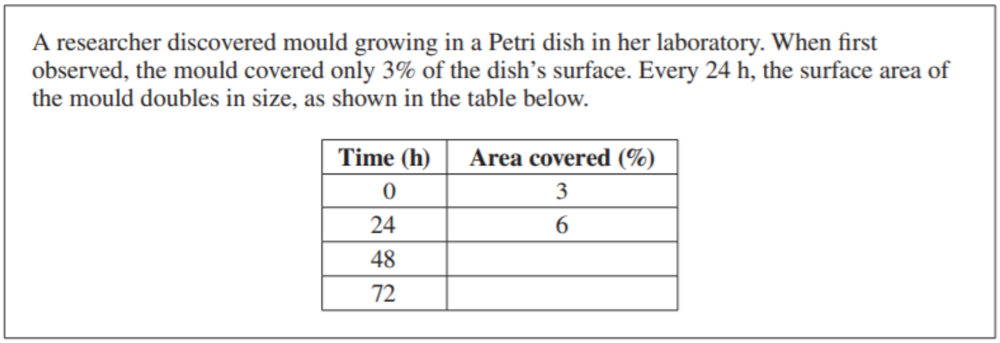
Careful to not put b as $\boldsymbol{0.3}$



A car dealership observed and recorded the value of a particular car over time. It was discovered that the value of the car is decreasing at an average rate of 18%/a. The initial value of the car was $29 300.

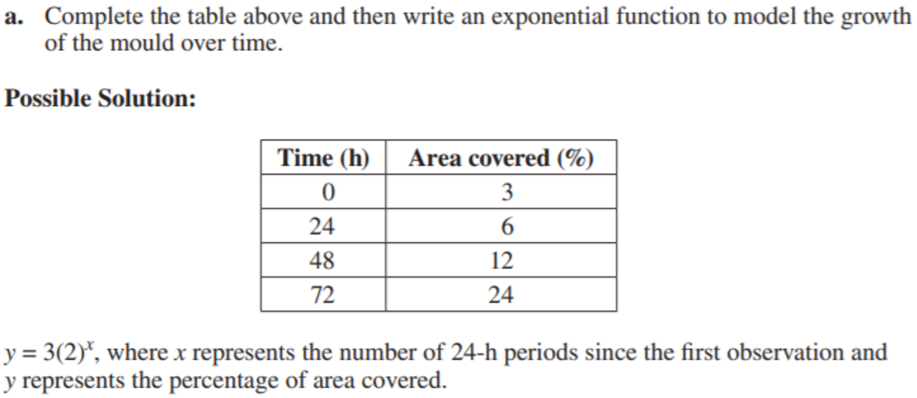
Which of the following exponential functions could be used to model the value of the car, $v(t)$, after t years?

A. $v(t) = 5\,274(0.82)^t$
B. $v(t) = 5\,274(1.18)^t$
*__C.__ $v(t) = 29\,300(0.82)^t$
D. $v(t) = 29\,300(1.18)^t$



A researcher discovered mould growing in a Petri dish in her laboratory. When first observed, the mould covered only 3% of the dish's surface. Every 24 h, the surface area of the mould doubles in size, as shown in the table below.

Time (h)	Area covered (%)
0	3
24	6
48	
72	



a. Complete the table above and then write an exponential function to model the growth of the mould over time.

Possible Solution:

Time (h)	Area covered (%)
0	3
24	6
48	12
72	24

$y = 3(2)^x$, where x represents the number of 24-h periods since the first observation and y represents the percentage of area covered.

Populations

Changes in populations, whether growth or decay, can be modeled using the formula $y = ab^x$ or $y = ab^{t/p}$, where:

- a represents the initial population
- b represents the multiplier
- x represents the unit time, if the period of time is 1
- t represents a length of time
- p represents a period of time, such a doubling period

EX: In 2001, the population of Sylvan Lake was 7500 people. If the population increased at a rate of 4.5% per year.

a) Write an exponential equation to model the population growth.

b) Determine the population in **2016**.

EX: The population of mallard ducks in Canada can be represented by $y = 1731000(0.95)^x$, where x is the number of years since 1975.

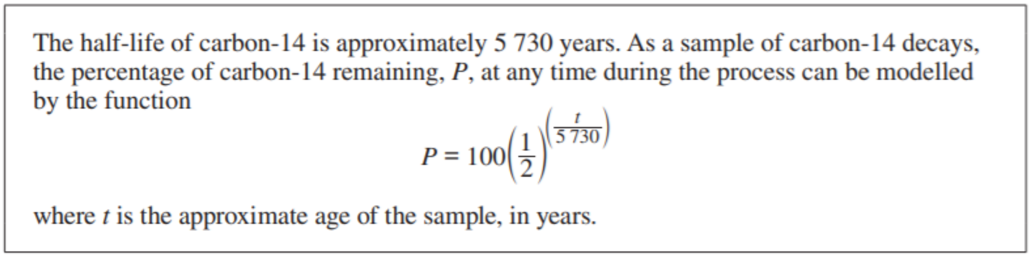
a) Is the population increasing or decreasing? Explain.

b) How many mallards were present in Canada in 1975?

c) How many years passed before one – half of the population remained?

Radioactive Decay/Half – Life

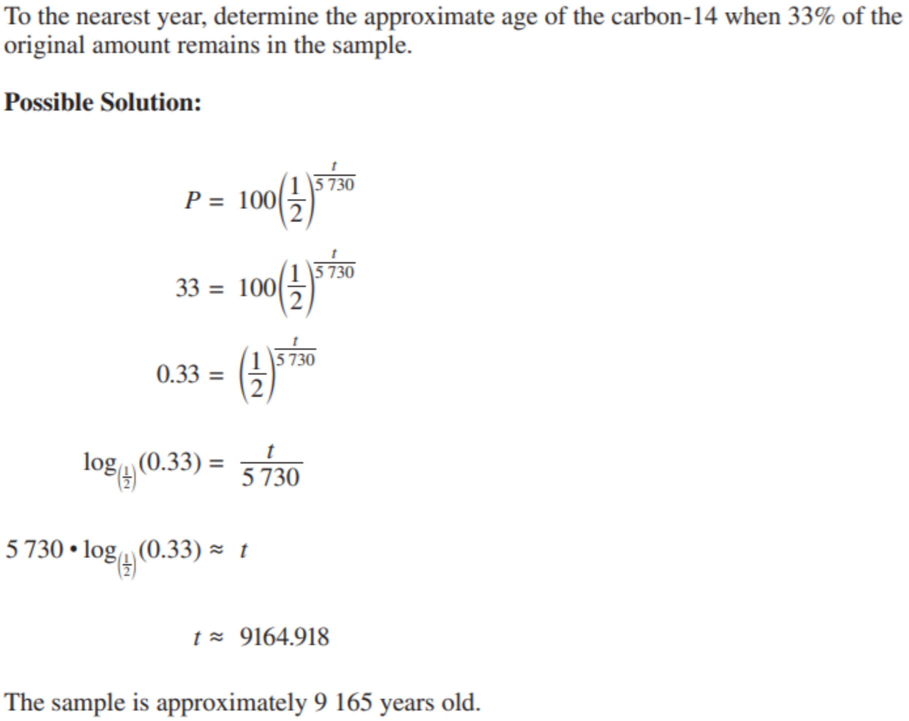
The same formula(s) can be used in this application as in population modeling: $y = ab^x$ or $y = ab^{t/p}$. Half – life is a representation of the amount of time it takes the radioactive particle to be reduced to half of its original amount (each time period sees the previous amount halved). This means that the multiplier, b, must be $\frac{1}{2}$



The half-life of carbon-14 is approximately 5 730 years. As a sample of carbon-14 decays, the percentage of carbon-14 remaining, *P*, at any time during the process can be modelled by the function

$$P = 100\left(\frac{1}{2}\right)^{\left(\frac{t}{5\,730}\right)}$$

where *t* is the approximate age of the sample, in years.



To the nearest year, determine the approximate age of the carbon-14 when 33% of the original amount remains in the sample.

Possible Solution:

$$P = 100\left(\frac{1}{2}\right)^{\frac{t}{5\,730}}$$

$$33 = 100\left(\frac{1}{2}\right)^{\frac{t}{5\,730}}$$

$$0.33 = \left(\frac{1}{2}\right)^{\frac{t}{5\,730}}$$

$$\log_{\left(\frac{1}{2}\right)}(0.33) = \frac{t}{5\,730}$$

$$5\,730 \cdot \log_{\left(\frac{1}{2}\right)}(0.33) \approx t$$

$$t \approx 9164.918$$

The sample is approximately 9 165 years old.

EX: Neptunium – 235 has a half – life of 396 days. A sample of neptunium – 235 has a mass of 250 g.

a) Write an exponential equation that models the amount, y, remaining after t days.

b) What mass of neptunium – 235 remains after 1000 days? Round to nearest tenth.

c) When will the sample be 50 g? Round answer to the nearest day.

IMPORTANT: If the question was **doubling** instead of halving, what would the b value be?

Compound Interest

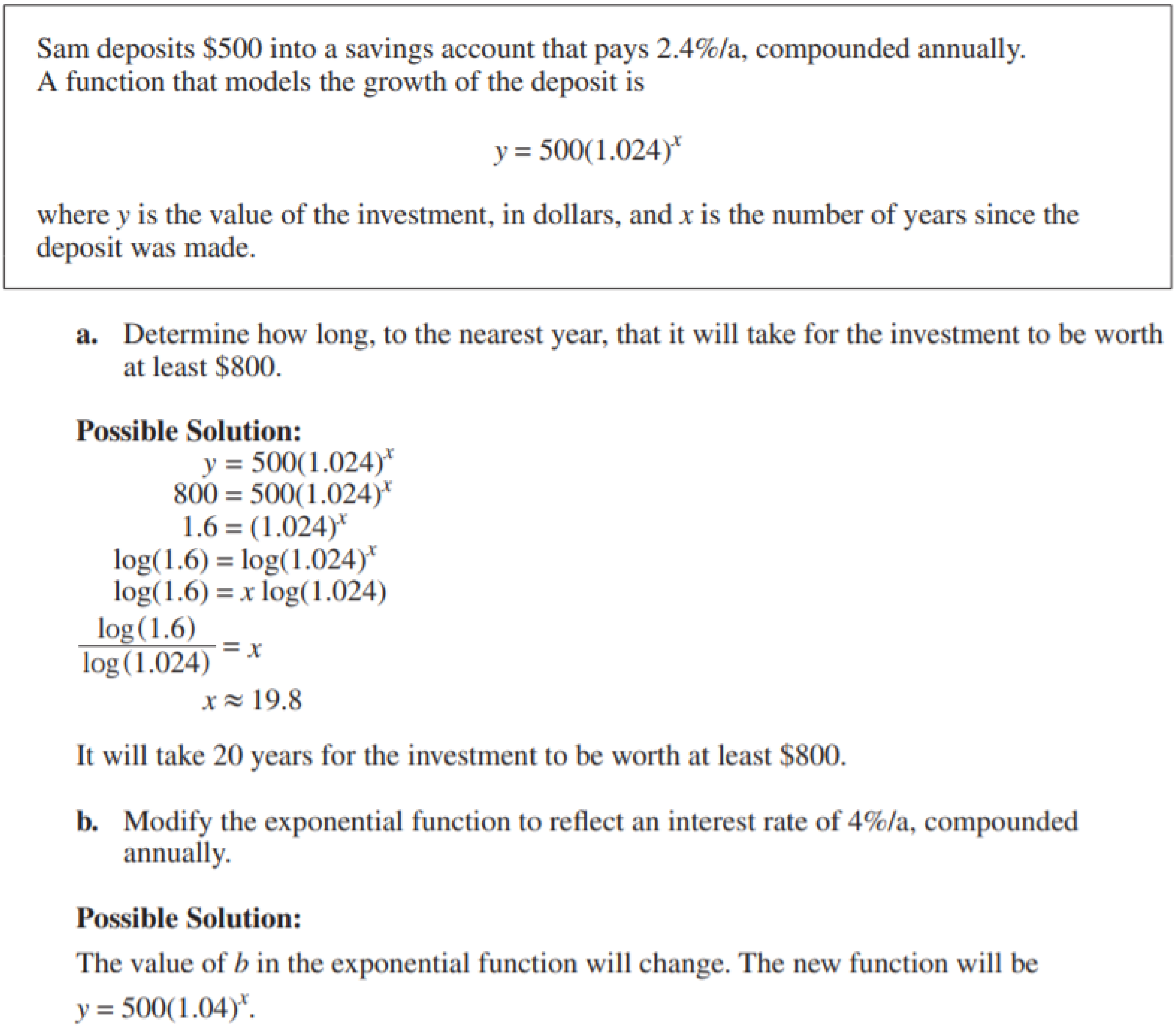
The formula for compound interest is a variation of the exponential growth/decay formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where A = amount of the investment at any time t
P = the initial value of the investment
r = the interest rate (as a decimal)
n = the number of compounds per year
t = the number of years

To determine the value of n, watch for the following words:

Compound Period	n
Annually	1
Semi – annually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365



Sam deposits \$500 into a savings account that pays 2.4%/a, compounded annually. A function that models the growth of the deposit is

$$y = 500(1.024)^x$$

where y is the value of the investment, in dollars, and x is the number of years since the deposit was made.

a. Determine how long, to the nearest year, that it will take for the investment to be worth at least \$800.

Possible Solution:

$$\begin{aligned} y &= 500(1.024)^x \\ 800 &= 500(1.024)^x \\ 1.6 &= (1.024)^x \\ \log(1.6) &= \log(1.024)^x \\ \log(1.6) &= x\log(1.024) \\ \frac{\log(1.6)}{\log(1.024)} &= x \\ x &\approx 19.8 \end{aligned}$$

It will take 20 years for the investment to be worth at least \$800.

b. Modify the exponential function to reflect an interest rate of 4%/a, compounded annually.

Possible Solution:

The value of b in the exponential function will change. The new function will be $y = 500(1.04)^x$.

EX: A student borrows \$2500 to further her post – secondary career. The bank is charging 6.5% interest, compounded semi – annually.

a) Write an exponential formula to represent the situation.

b) If it takes the student 5 years before she can repay the loan, what amount will she be required to pay?

c) How much interest will she pay on the loan if she pays all she owes in one payment 5 years after she takes out the loan?

The Richter Scale

The Richter Scale is a logarithmic scale used to describe the relative strengths of earthquakes. The formula for determining the magnitude of earthquakes is:

$$I = I_o 10^M$$

Where M is the magnitude of the earthquake, I is the intensity of the earthquake, and I_0 is the minimum intensity an earthquake can have.

EX: How many times as intense is an earthquake of magnitude 8 than magnitude 6 on the Richter Scale?

- To determine how many times bigger one number is than the other, we divide them

$$\frac{I_8}{I_6} = \frac{I_o 10^8}{I_o 10^6}$$

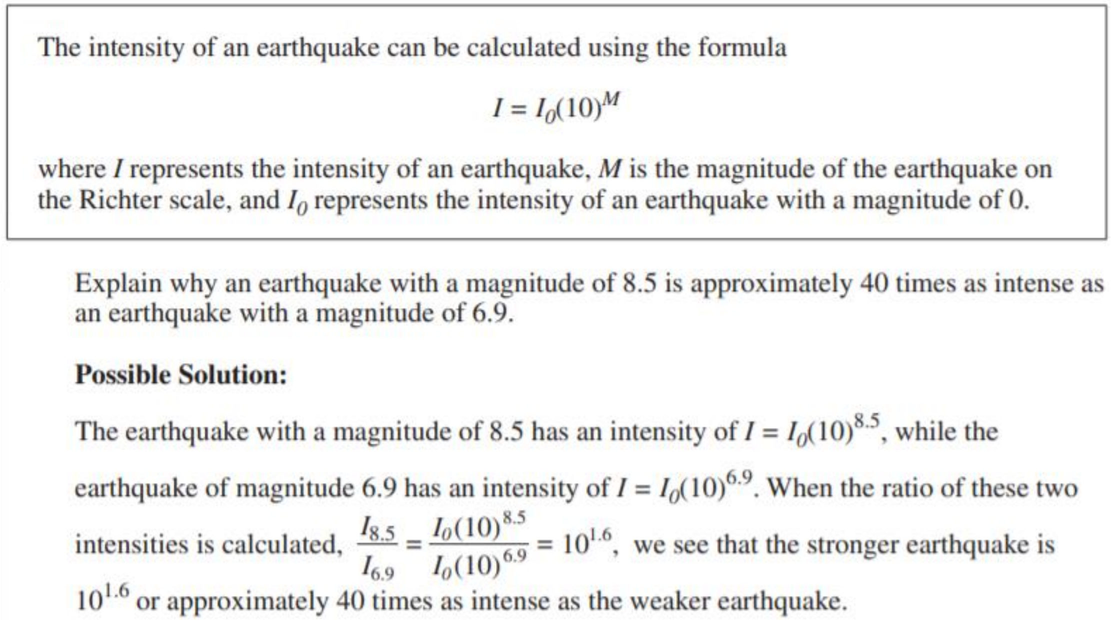
- The I_o cancels for both!

$$\frac{I_8}{I_6} = \frac{10^8}{10^6}$$

- We subtract exponents when dividing with the same base

$$\frac{I_8}{I_6} = 10^2 = \mathbf{100}$$

The intensity is 100 times more!



The intensity of an earthquake can be calculated using the formula

$$I = I_0(10)^M$$

where I represents the intensity of an earthquake, M is the magnitude of the earthquake on the Richter scale, and I_0 represents the intensity of an earthquake with a magnitude of 0.

Explain why an earthquake with a magnitude of 8.5 is approximately 40 times as intense as an earthquake with a magnitude of 6.9.

Possible Solution:

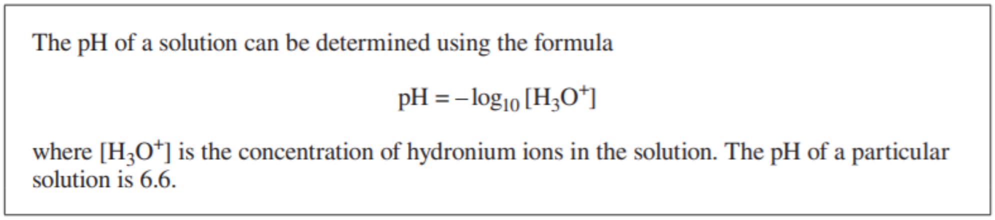
The earthquake with a magnitude of 8.5 has an intensity of $I = I_0(10)^{8.5}$, while the earthquake of magnitude 6.9 has an intensity of $I = I_0(10)^{6.9}$. When the ratio of these two intensities is calculated, $\frac{I_{8.5}}{I_{6.9}} = \frac{I_0(10)^{8.5}}{I_0(10)^{6.9}} = 10^{1.6}$, we see that the stronger earthquake is $10^{1.6}$ or approximately 40 times as intense as the weaker earthquake.

EX: The strongest earthquake in Alberta was 4.4. How much stronger is the strongest earthquake in the World if it measured 9.5?

The pH Scale

pH is a measure of the concentration of hydrogen ions, $[H^+\ (aq)]$, in a solution. The greater the concentration of hydrogen ions, the lower the pH. A pH between 0 and 7 refers to an acid, a pH of 7 is a neutral solution and a pH between 7 and 14 refers to a basic solution. pH can be found using the following formula:

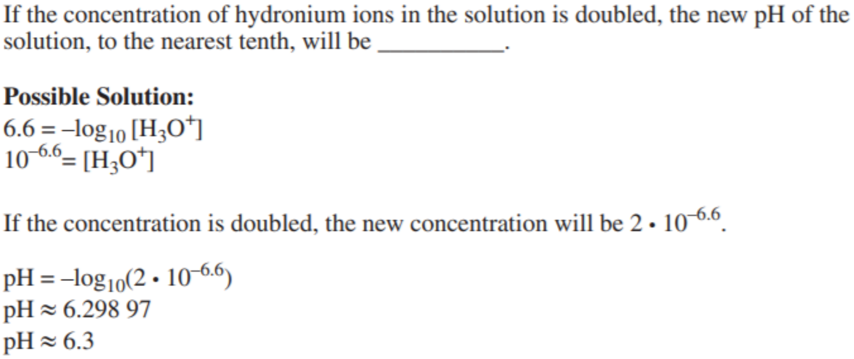
$$pH = -log[H^+(aq)]$$



The pH of a solution can be determined using the formula

$$\text{pH} = -\log_{10}[\text{H}_3\text{O}^+]$$

where $[\text{H}_3\text{O}^+]$ is the concentration of hydronium ions in the solution. The pH of a particular solution is 6.6.



If the concentration of hydronium ions in the solution is doubled, the new pH of the solution, to the nearest tenth, will be ___________.

Possible Solution:

$6.6 = -\log_{10}[\text{H}_3\text{O}^+]$

$10^{-6.6} = [\text{H}_3\text{O}^+]$

If the concentration is doubled, the new concentration will be $2 \cdot 10^{-6.6}$.

$\text{pH} = -\log_{10}(2 \cdot 10^{-6.6})$

$\text{pH} \approx 6.298\ 97$

$\text{pH} \approx 6.3$

EX: In spring, the pH of the water in a mountain stream is 4.2. It is 90.5 times more acidic than in the fall. Determine the pH of the water in the fall.

Decibel Level

The intensity of sound is measured in decibels (dB). The formula for the loudness of sound is:

$$L = 10log\left(\frac{I}{I_0}\right)$$

Where L is the level of sound (dB), I is the intensity of the sound, $I_0$is the minimum sound the human ear can detect.

EX: Portable music players can produce sounds up to 125dB. Any sound about 90dB can cause hearing loss if exposure is prolonged. To be safe, experts suggest that the volume does not exceed 60% of the maximum that the player can produce. Assuming your player is at this maximum, how many times more intense is this than the recommended setting?

Homework

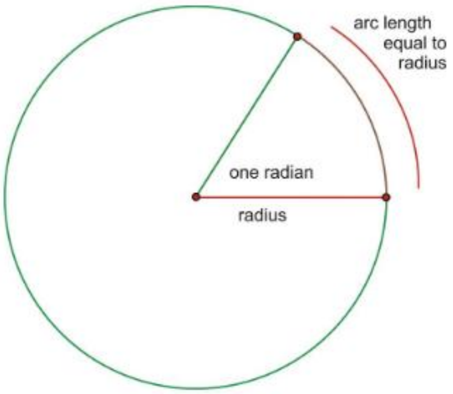
Unit 7- Sinusoidal Functions

Lesson 25- Graphs of Sinusoidal Functions

Measure of Radians

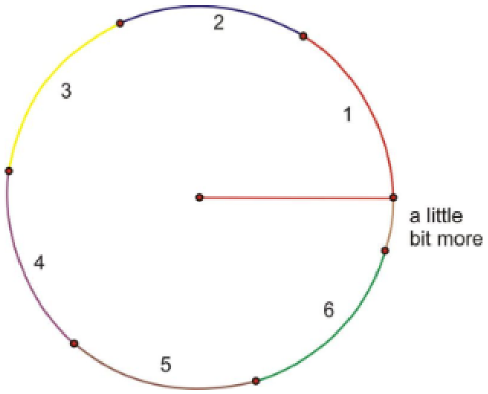
Until now, we have used degrees to measure angles. But, what exactly is a degree?

- A degree is $\frac{1}{360}th$ of a complete rotation around a circle.
- Radians are alternate units used to measure angles in trigonometry. Just as it sounds, a radian is based on the radius of a circle. **One radian** (abbreviated **rad**) is the angle created by bending the radius length around the arc of a circle. Because a radian is based on an actual part of the circle rather than an arbitrary division, it is a much more natural unit of angle measure for upper level mathematics.


arc length
equal to
radius
one radian
radius

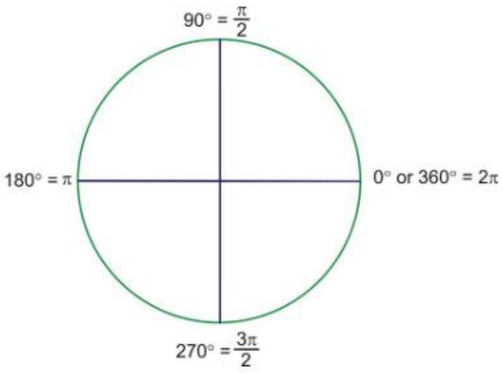
What if we were to rotate all the way around the circle?

- Continuing to add radius lengths, we find that it takes a little more than 6 of them to complete the rotation.
- Recall from geometry that the arc length of a complete rotation is the circumference, where the formula is equal to 2π times the length of the radius. 2π is approximately 6.28, so the circumference is a little more than 6 radius lengths. Or, in terms of radian measure, a complete rotation (360 degrees) is 2π radians.

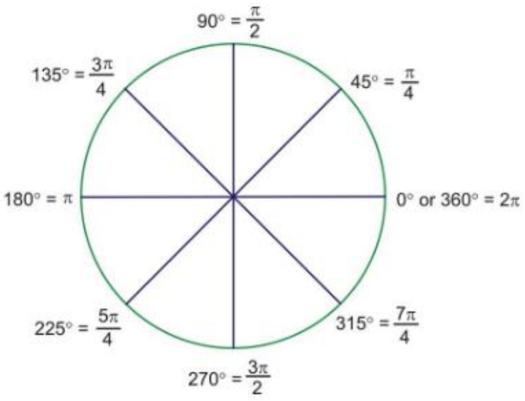

1
2
3
4
5
6
a little
bit more

360 degrees=2π radians

- With this as our starting point, we can find the radian measure of other angles. Half of a rotation, or 180 degrees, must therefore be π radians, and 90 degrees must be $\frac{1\pi}{2}$, written 2. Extending the radian measure past the first quadrant, the quadrantal angles have been determined, except 270°. Because 270°is halfway between 180° (π) and 360° (2π), it must be 1.5π, usually written $\frac{3\pi}{2}$.


90° = $\frac{\pi}{2}$
180° = π
0° or 360° = 2π
270° = $\frac{3\pi}{2}$

- Other common radian measure of angles are found in a similar manner.

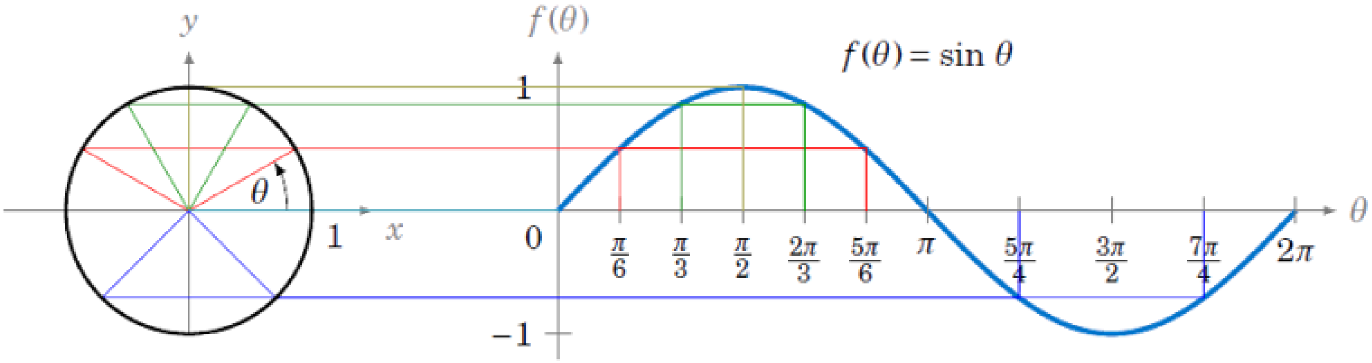

90° = $\frac{\pi}{2}$
135° = $\frac{3\pi}{4}$
45° = $\frac{\pi}{4}$
180° = π
0° or 360° = 2π
225° = $\frac{5\pi}{4}$
315° = $\frac{7\pi}{4}$
270° = $\frac{3\pi}{2}$

You are not required to convert between degrees and radians in Math 30-2. However, you should have a general understanding of the size of angles measured in radians.

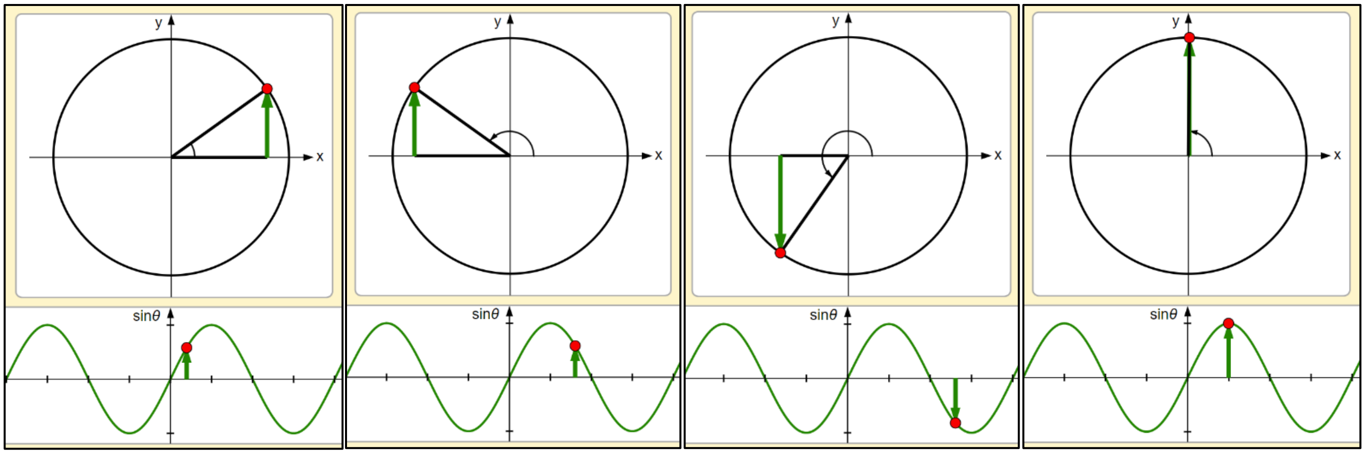
Sinusoidal Graphs

Objects moving in circular motion can be graphed using a sinusoidal function.

- In the picture below, the y-axis represents the height of the object going in a circle, while the x-axis represents the angle of the object measured in radians.
- The equation can be modified so that the x-axis is represented by a different unit of measurement, such as time or distance.


y
x
1
θ
$f(\theta)$
$f(\theta) = \sin\theta$
1
0
−1
$\frac{\pi}{6}$
$\frac{\pi}{3}$
$\frac{\pi}{2}$
$\frac{2\pi}{3}$
$\frac{5\pi}{6}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π
θ

The equation of the graph above is $y = sin(x)$ or $y = sin(\theta)$, where θ is the given angle. But what does $y = sin(x)$ even mean? Taking the sine of the angle gives the height at the given point on the circle. The 4 pictures below show how the height on the circle relates to the height on the graph.

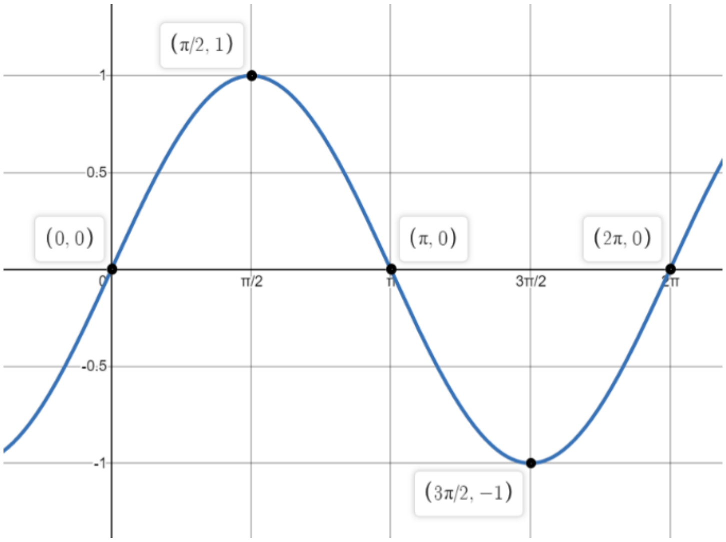

y
x
sinθ
y
x
sinθ
y
x
sinθ
y
x
sinθ

To the right is the graph of $y = sin(x)$

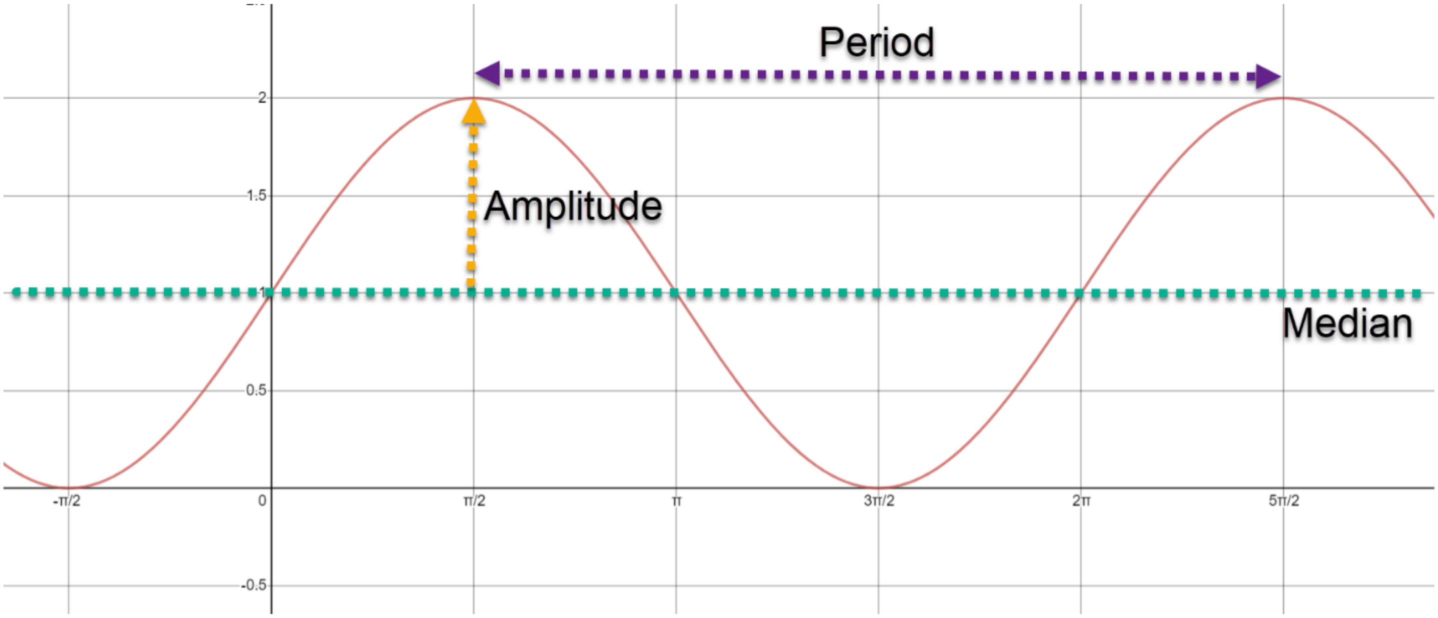
However, circular motion in real-world situations do not always follow this simple graph. The graph can be transitioned up, down, left or right. As well as stretched, or compressed, horizontally or vertically.

The equation that includes all these transformations is shown below.

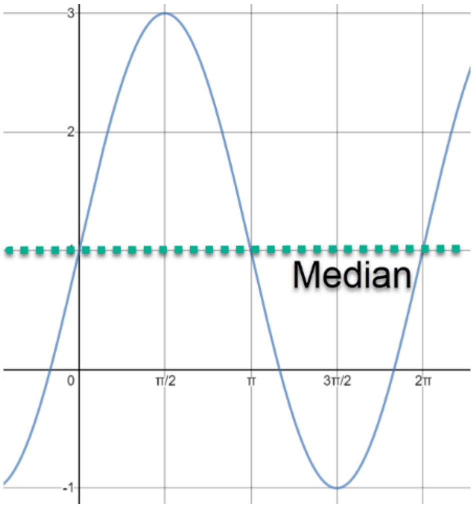
$$y = sin(x)$$



$$y = a \sin\left(bx - c\right) + d$$



"d"



The "d" value in the equation above represents the **median** value, or sometimes called the midline value.

- The median for the graph of $y = sin\ (x)$ is the line $y = 0$.
- If the entire graph is moved up by 1, then the median would be the line $y = 1$.
    - You can see from the graph, that "d" is how much the graph has been shifted up or down.

Sometimes you will need to determine the median ("d") in a given situation.

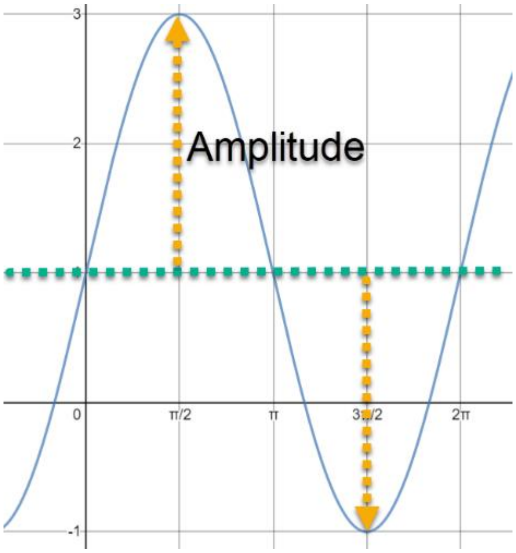
- The median is the *average* between the maximum and the minimum y-values.
- In the example to the right, the maximum is 3 and the minimum is -1.
    - To find the average between 2 numbers, we add the numbers together and divide by 2. This can be written as

$$d = \frac{max + min}{2}$$ ← This is not on your formula sheet

$$d = \frac{3 + (-1)}{2}$$

$$\boldsymbol{d = 1}$$

"a"



The "a" value in the equation above represents the **amplitude** of the graph which is also considered the vertical stretch.

- The amplitude is the positive distance from the midline to the max or min value.

To find the amplitude ("a") we will first find the height of the entire function by subtracting the min value from the max value. In the example to the right, this would be $3 - (-1)$, which is equal to 4.

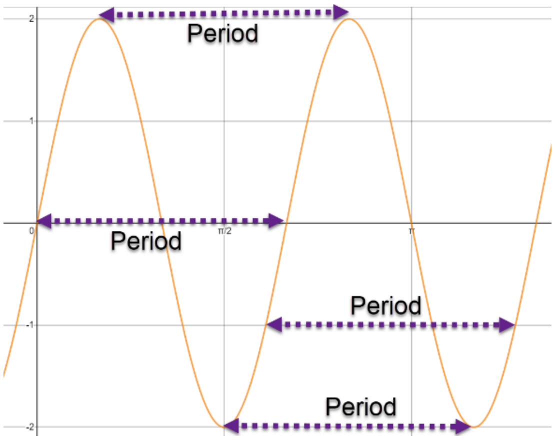
- So, the height of our graph, from top to bottom, is 4.
- The ***amplitude is just half of the total height***, which is 2. This can be written as

$$a = \frac{max - min}{2}$$ ← This is not on your formula sheet

$$a = \frac{3 - (-1)}{2}$$

$$\boldsymbol{a = 2}$$

"b"



The "b" value in the equation above is related to the **period** of the function.

- The period of a sine function is angle or time to complete one full cycle.

The "b" value is not the period

"b" is the number of times the graph repeats itself in 2π radians.

- Once you know the period of a function, "b" can be found using the formula

$$period = \frac{2\pi}{b}$$ ← This **is** on your formula sheet

We can rearrange this equation to get

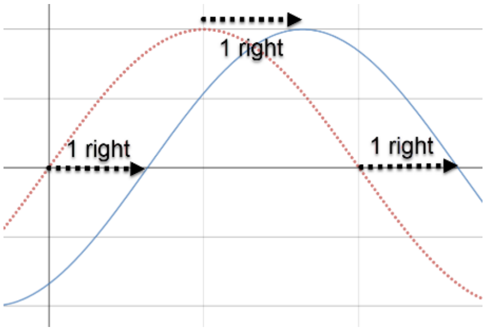
$$b = \frac{2\pi}{period}$$

If the period from the example to the right is $\frac{2\pi}{3}$,

$$b = \frac{2\pi}{\left(\frac{2\pi}{3}\right)}$$

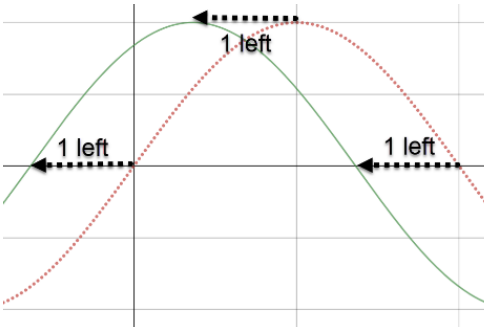
$$\boldsymbol{b = 3}$$

"c"



The "c" value in the equation above is related to the **horizontal shift** or **phase shift** of the graph.

- In the example to the right, the red dotted line is the graph $y = sin\ (x)$ and the blue solid graph is $y = sin\ (x - 1)$. In this case, "c" is equal to 1.
    - This means the graph will shift 1 to the right. You may think "c" would be -1 but remember that the equation is $y = asin\ (bx - c)\ + d$.
    - The "minus" sign in the equation makes "c" the opposite of the way it looks in the equation. Everything will shift **1 to the right**.
- In the next example, the green solid line is $y = sin\ (x + 1)$.
    - Again, "c" will be the opposite of the way it looks in the equations. "c" is -1 and everything will shift **1 to the left**.



****In Math 30-2, you will never be required to determine the value of "c", but you should have a general understanding that if $c \neq 0$, then a phase shift has occurred.****

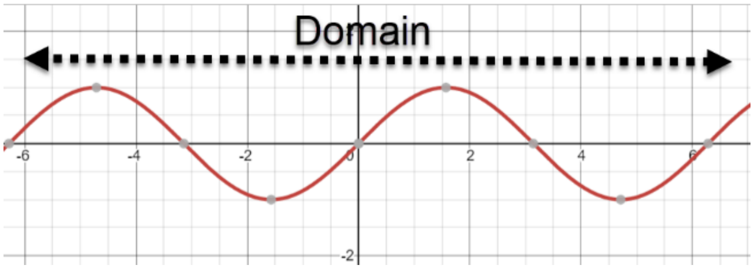
Domain and Range

The **domain** of a sinusoidal function is

$$\boldsymbol{Domain}: x|\ x \in R$$

Or

$$x \in R$$



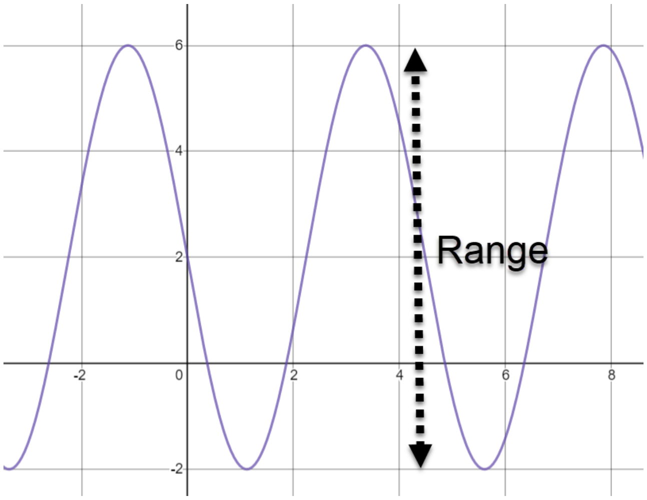
The **range** of a sinusoidal function is the y-values between the maximum and minimum values.

- The range of the example to the right is

$$\boldsymbol{Range}: y| -2 \leq y \leq 6, y \in R$$

Or

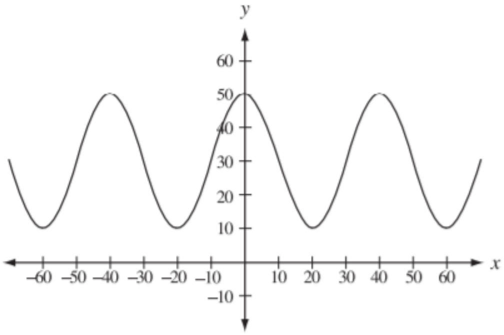
$$-2 \leq y \leq 6$$



IMPORTANT: The **practical domain and range** of an application question may be different. You will need to interpret the proper domain and range for the given situation. See the example questions below.

EX:

The graph of a sinusoidal function is shown below.



Possible values for the amplitude and median of the sinusoidal function are 10, 20, 30, 40, and 50.

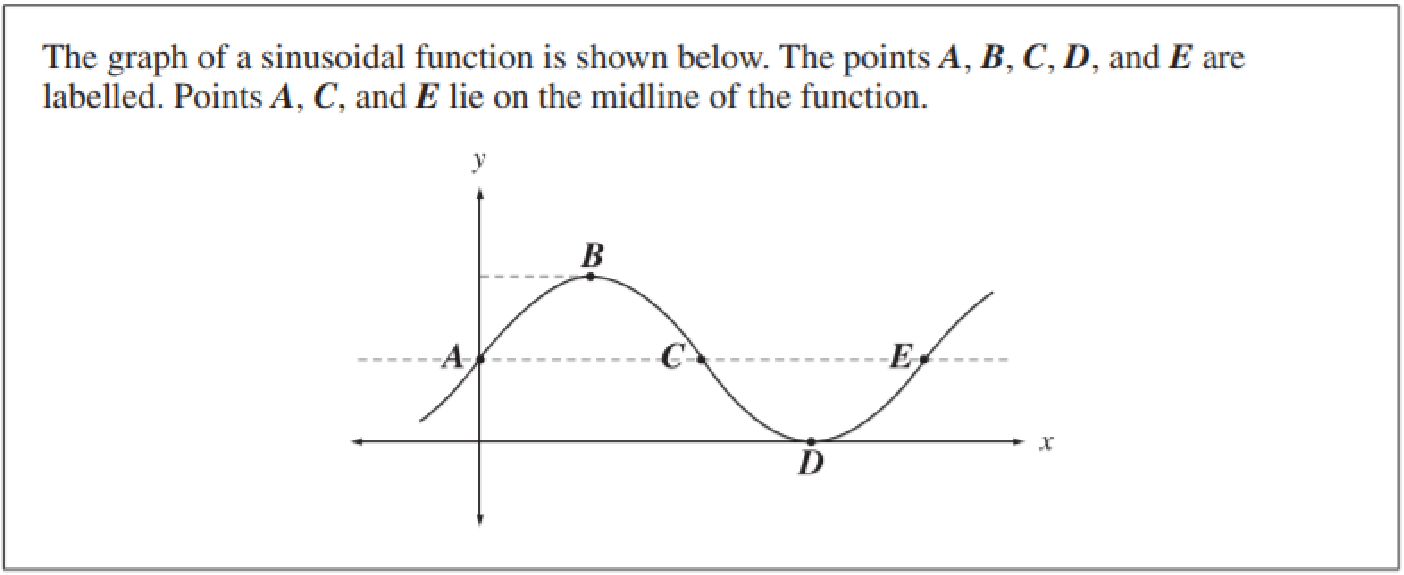
Record the values of the amplitude and the median of the sinusoidal function above in the blanks.

Value:	______	______
Characteristic:	**Amplitude**	**Median**

Solution:

Value:	20	30
Characteristic:	**Amplitude**	**Median**

EX:



The graph of a sinusoidal function is shown below. The points ***A***, ***B***, ***C***, ***D***, and ***E*** are labelled. Points ***A***, ***C***, and ***E*** lie on the midline of the function.

a. Mary says that in order to find the period of the function, she would need to know the coordinates of points A and E. Bill says that he could find the period using the coordinates B and D. Both Mary and Bill are correct. Explain why.

- Mary is correct because the horizontal distance between A and E would be the period, as it is the length of time it takes the function to complete one cycle.
- Bill is also correct, as the horizonal distance between B and D represents half the period. He would need to remember to double this result to determine the period.

b. Select all points that represent the x-intercepts of the function.

- D is the only x-intercept visible on the graph above.

c. Select all points that represent the minimum value of the function.

- D is the only minimum point visible on the graph above.

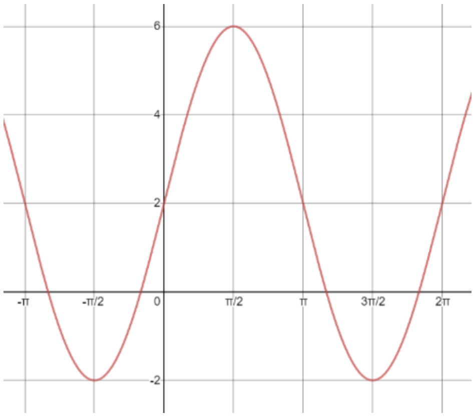
d. Select two points that could be used to determine the amplitude of the function. Describe a process that could be used to determine the amplitude using the two selected points.

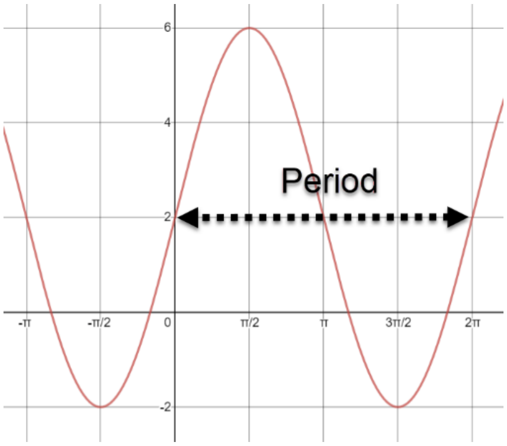
- Select B and C and find the vertical distance between them by subtracting the y-coordinate of Point C from the y-coordinate of Point B. This distance would represent the amplitude.

EX: Determine the median line, amplitude, period of the following graphs. Also determine if the graph has been phase shifted.

$$y = 4\,sin\,(x)\ + 2$$

- Median or “d” value: You can see that the midline of the graph is now at +2. You can also see from the equation that “d” is +2. The median line is the line y=2.
- Amplitude: The amplitude is the “a” value in the question which is 4. You can also see this from the graph. If the median line is at 2, and the max is 6, then the amplitude is 4.





- Period: The period can be directly seen from the graph. Look for where pattern of the graph repeats. The graph repeats after 2π radians. This could also be found by looking at the equation y=4sin x +2. The "b" value is equal to 1 in this case. To find the period, we use

$$period = \frac{2\pi}{b}$$

$$period = \frac{2\pi}{1}$$

$$period = 2\pi$$

- Phase Shifted? From the equation $y = 4sin\ (x)\ + 2$ we can see that there is no "c" value. Therefore, is has not been phase shifted.

EX: Create a sinusoidal function based on the following descriptions:

1. Amplitude of 5 units, period of 6π radians, and a midline equation of $y = 2$

2. Amplitude of 9 units, period of 0.25 radians, and a midline equation of $y = -3$

EX: Determine the amplitude, period, equation of the midline, maximum and minimum values, and the domain and range for the following function:
$y = 4.32\sin(12.34x + 3.21) - 9.12$ If required, rounded to the nearest hundredth.

Homework

Lesson 26- Applications of Sinusoidal Functions

Word Problems:

The values of "a", "b", "c" and "d" from the equation $y = \boldsymbol{a}\sin(\boldsymbol{b}x - \boldsymbol{c}) + \boldsymbol{d}$, will not always be given directly.

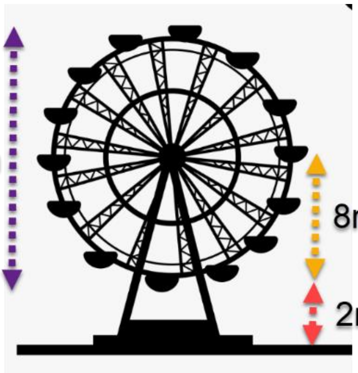
- Often you will be required to infer aspects of the graph based on the information you are given.
- Again, you will not be required to determine the value of "c".

EX:

A Ferris wheel has a radius of 8 m and its centre is 10 m above the ground. A rider gets on a chair of the Ferris wheel at its lowest point and completes one full revolution in 48 s.

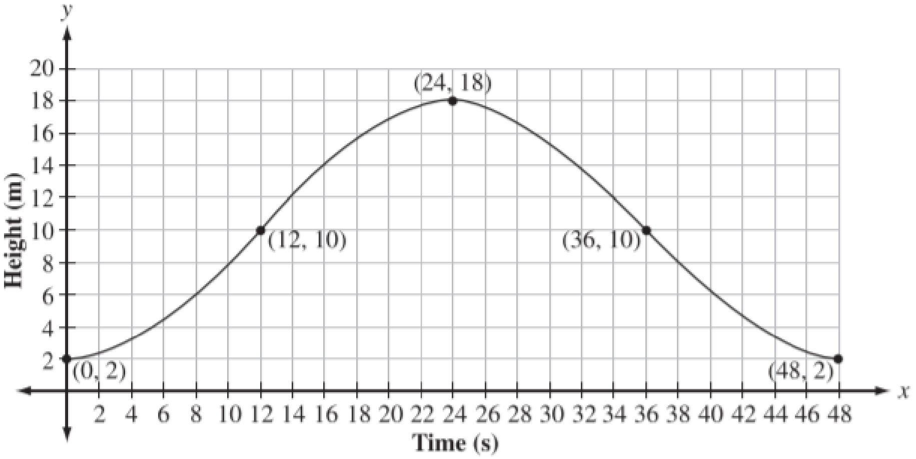
a. Sketch a graph on the grid below to show the height of the rider above the ground, *y*, over time, *x*, for the first 48 s. Label key points on the graph.

- The question says that the radius of the Ferris wheel is 8m and its center is 10m from the ground. The lowest point must be 2m from the ground which is our starting point.
  - **Plot the point (0,2).**
- The ride completes one full revolution in 48s. That means the height will be back down to 2m after 48s.
  - **Plot the point (48,2).**
- Halfway between the start and the finish, the ride will be at its highest point. Halfway between 0 and 48 seconds is 24 seconds. The height will be 2m plus the diameter of the Ferris wheel which is 16m. The max height is 18m.
  - **Plot the point (24,18).**



You now have enough info to make a sketch of the graph.

Possible Solution:



Notice the points (12,10), and (36,10) on the graph above. These points occur right between the minimum and maximum values and have a height of 10m. What would these points represent?

b. State the amplitude, period, and equation of the midline for the function sketched in part a, on the pervious page.

- The amplitude represents the radius of the Ferris wheel, which is 8m
- The period is 48s.
- The midline represents the heights of the centre of the Ferris wheel above the ground. The equation of the midline is $y = 10$

c. Determine a function of the form $y = a \cdot \sin(bx - 1.57) + d$, where y represents the height of the rider above the ground and x represents the time after the ride has started, that could be used to model the height above the ground of a rider on the Ferris wheel described above.

- Notice in the equation that the value of "c" is already given. We just need to determine a, b and d.
    - "a" is amplitude which is 8m
    - "d" is the midline which is 10m
    - "b" can be found using the period and the equation

$$b = \frac{2\pi}{period}$$

$$b = \frac{2\pi}{48}$$

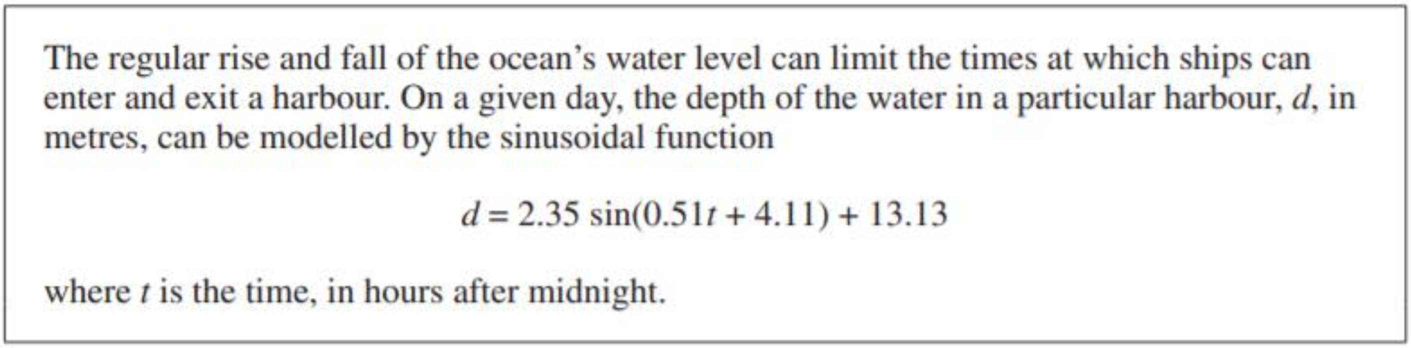
$$b \approx 0.13$$

- Put it all together

$$y = a\sin(bx - 1.57) + d$$

$$\boldsymbol{y = 8sin\ (0.13x - 1.57)\ + 10}$$

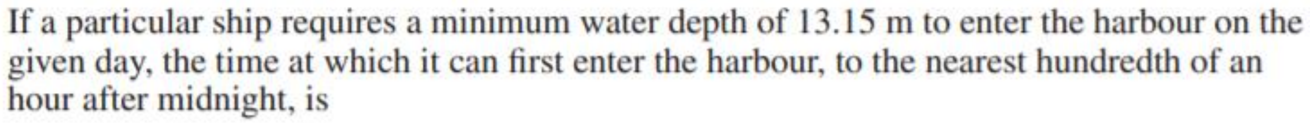
EX:



The regular rise and fall of the ocean's water level can limit the times at which ships can enter and exit a harbour. On a given day, the depth of the water in a particular harbour, *d*, in metres, can be modelled by the sinusoidal function

$$d = 2.35\sin(0.51t + 4.11) + 13.13$$

where *t* is the time, in hours after midnight.



If a particular ship requires a minimum water depth of 13.15 m to enter the harbour on the given day, the time at which it can first enter the harbour, to the nearest hundredth of an hour after midnight, is

- The equation above represents a sinusoidal function, where the water level is rising and falling. We need to determine the time when the distance is at 13.15 m. We will use technology to find x when y is 13.15.
    - Graph the equation

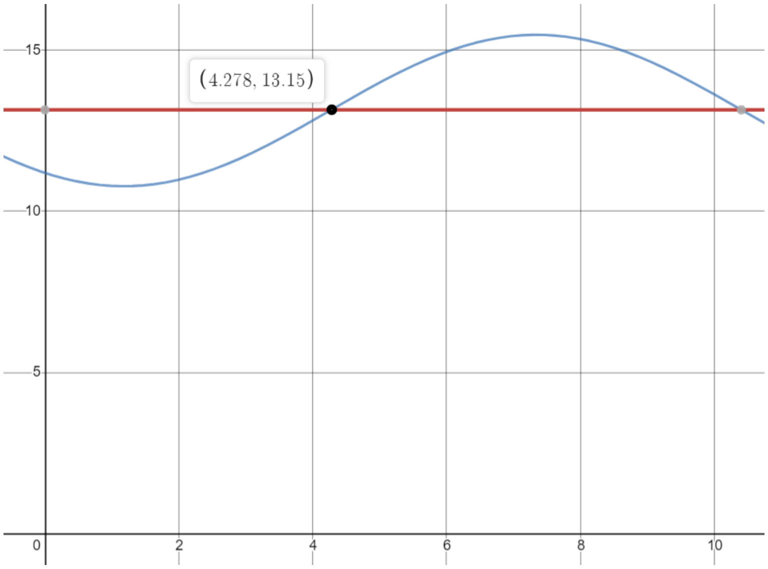
$$y = 2.35sin\ (0.51x + 4.11)\ + 13.13$$

-
    - Graph the line

$$y = 13.15$$

- Find the intersection

The time that the ship can first enter the harbour is at **4.28 hours**.



EX: The London Eye has a diameter of 122 m and it is 135 m from the ground to the top. It takes 30 minutes for one rotation of the wheel. Passengers board at the bottom of the ride, and the wheel moves slowly enough that it is not necessary to stop to let passengers on/off.

a) Write a sinusoidal function the represents the height of a passenger riding the London Eye. Over what domain is the function valid?

b) How high above the ground would a passenger be after being on the ride for one – third of its duration?

c) At what time(s) would the rider be 100 m above the ground?

EX: The depth, d meters, of water in a harbour, t hours after midnight, can be approximated by the function $d(t) = 12 + 5cos0.5t$, where $0 \leq t \leq 24$.

a) Determine the maximum and minimum depths of water in the harbour.

b) Determine the period of the function

c) Write a suitable window which can be used to display the graph of the function.

d) What is the depth of the water, to the nearest tenth of a meter at 2:00 am?

e) A ship which requires a minimum of 8.5 meters of water is in harbour at midnight. By what time, to the nearest minute, must it leave to prevent grounding?

f) What is the next time, to the nearest necessary minute, that the ship can return to the harbour?

EX: The tides in a particular bay can be modeled with an equation of the form $d = \text{a}\cos(bt) + c$, where t represents the number of hours since high-tide and d represents the depth of water in the bay. The maximum depth of water is 36 ft, the minimum depth is 22 feet and high-tide is hit every 12 hours.

a) Determine the values of a, b and c in the model.

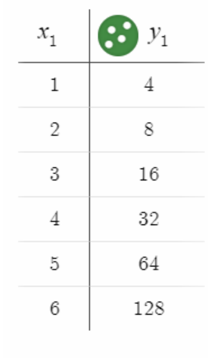
b) Tanker boats cannot be in the bay when the depth of water is less or equal to 25 ft. Set up an inequality and solve it graphically to determine all points in time, t, on the interval $0 \leq t \leq 24$ when tankers cannot be in the bay. Round to the nearest tenth of an hour.

Homework

Unit 8- Regressions

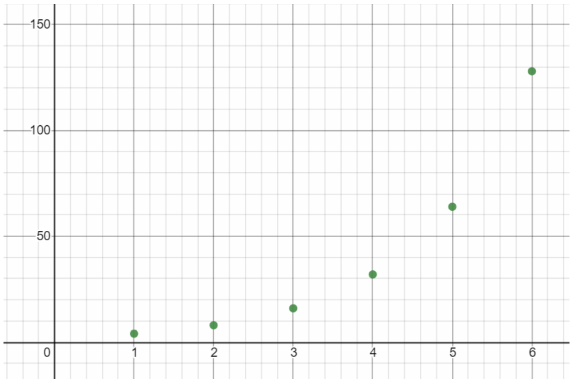
In this unit you will be using your graphing calculator to determine a function that is the "best fit" for the data provided.

For example, what is the function for the data below?



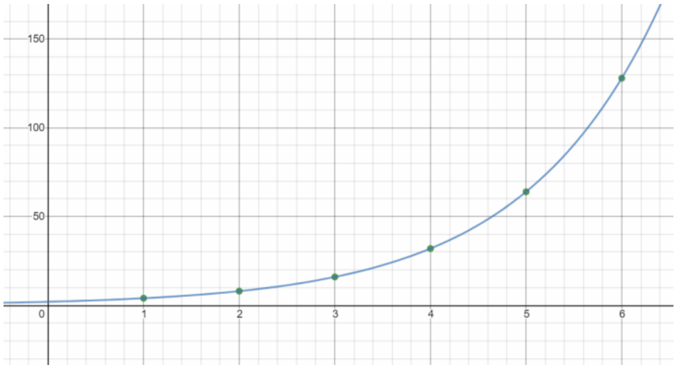
x_1	y_1
1	4
2	8
3	16
4	32
5	64
6	128

It becomes clearer when we plot the data on a graph.



This appears to follow an exponential function. $y = a(b)^x$

It is even more clear when we use technology to create a line of best fit.

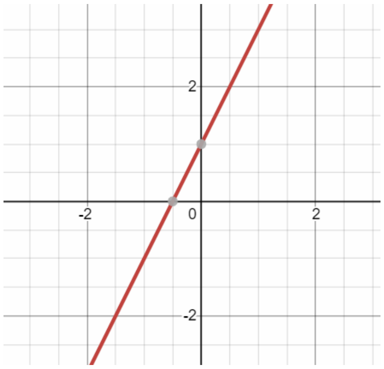


In Math 30-2, we have explored the following types of functions.

You need to look for these patterns.

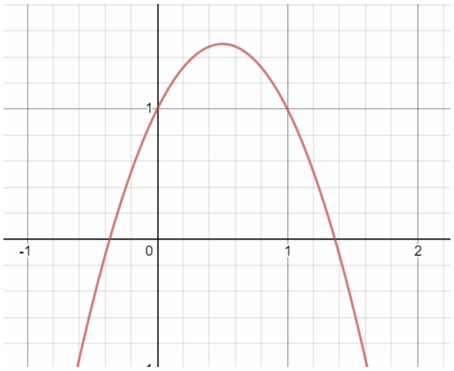
Linear

$$y = mx + b$$



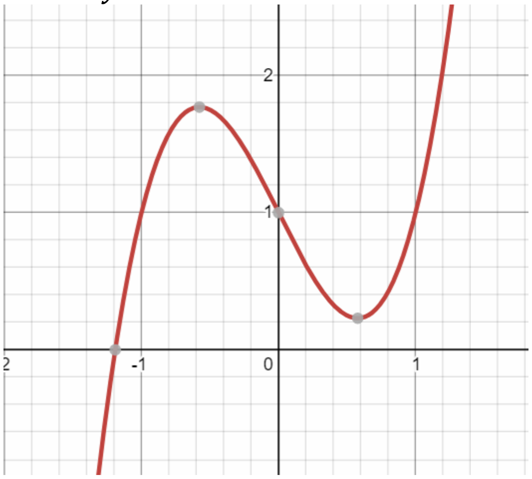
Quadradic

$$y = ax^2 + bx + c$$



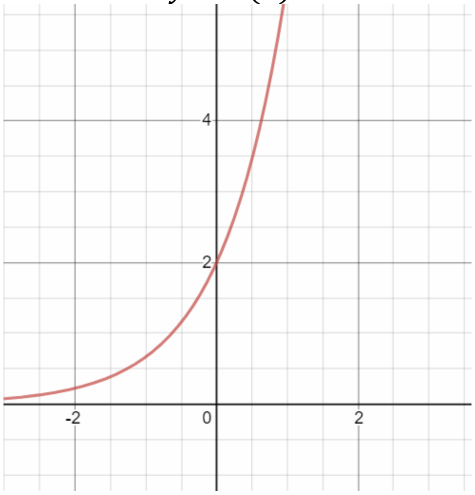
Cubic

$$y = ax^3 + bx^2 + cx + d$$



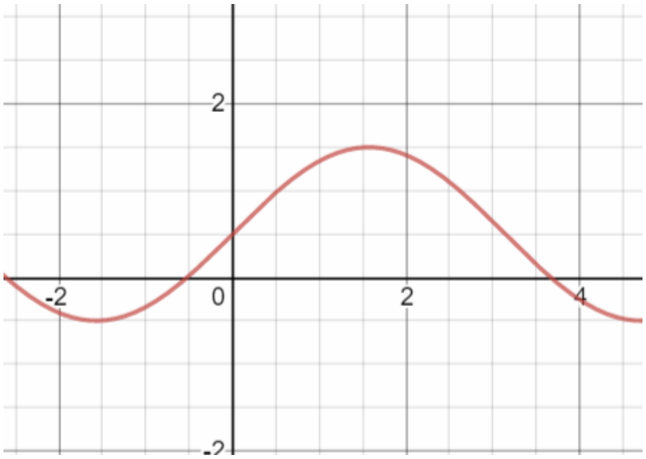
Exponential

$$y = a(b)^x$$



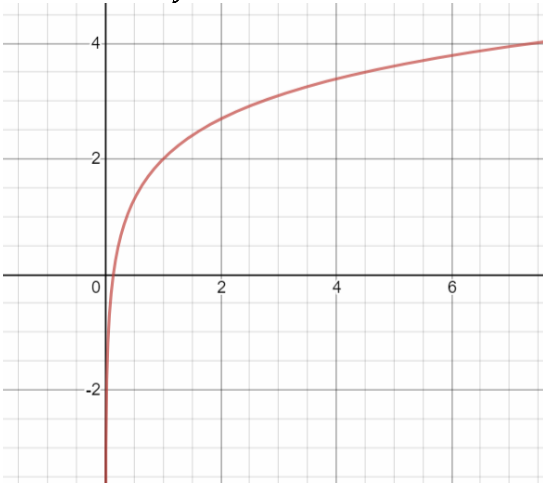
Sinusoidal

$$y = a\sin(bx + c) + d$$



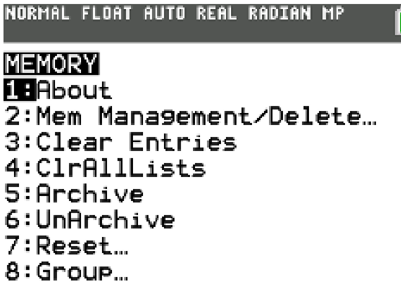
Logarithmic

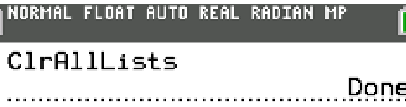
$$y = a + b \cdot lnx$$



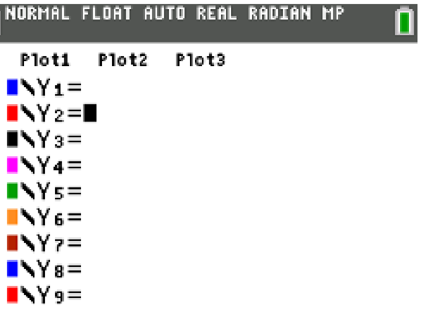
Graphing a Regression Equation with the TI – 84 Family

1. Turn on your calculator. Press 2ND + to get the leftmost picture directly below, then select **4: ClrAllLists** and press enter to clear all lists. Press ENTER to get the middle screen below. Also, press Y= and clear this menu, as shown in the rightmost picture below.

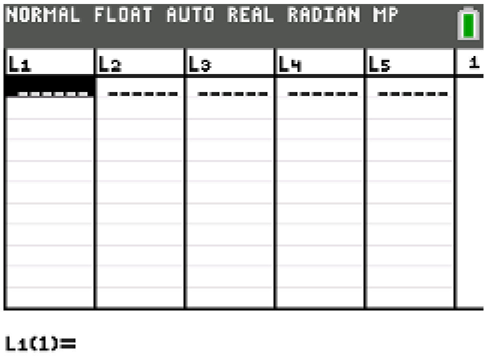

NORMAL FLOAT AUTO REAL RADIAN MP
MEMORY
1:About
2:Mem Management/Delete…
3:Clear Entries
4:ClrAllLists
5:Archive
6:UnArchive
7:Reset…
8:Group…


NORMAL FLOAT AUTO REAL RADIAN MP

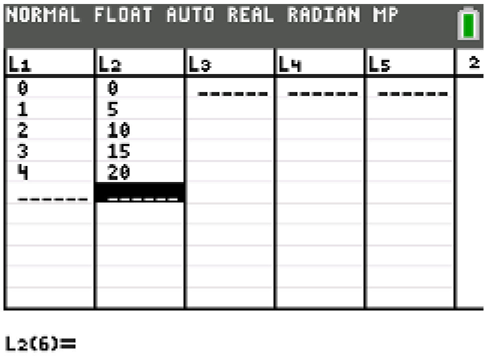
ClrAllLists
Done


NORMAL FLOAT AUTO REAL RADIAN MP
Plot1 Plot2 Plot3
Y_1=
Y_2=
Y_3=
Y_4=
Y_5=
Y_6=
Y_7=
Y_8=
Y_9=

2. Now the calculator is ready, so time to input your data! Press STAT, then **1: Edit**. A number of columns will appear on the screen, as shown. Put your *x* – data in the column under **L₁** and your *y* – data in the column under **L₂**. Press the arrow keys to move from column **L₁** to column **L₂** and back. Please enter the data shown in the picture below on the right.


NORMAL FLOAT AUTO REAL RADIAN MP
L1
L2
L3
L4
L5
1

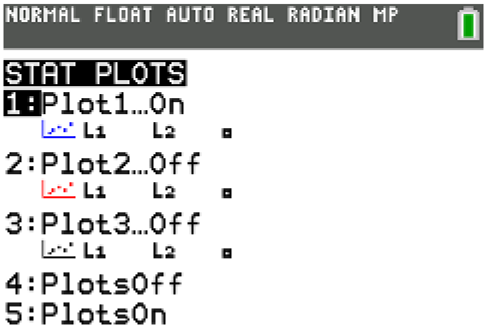
L1(1)=


NORMAL FLOAT AUTO REAL RADIAN MP
2

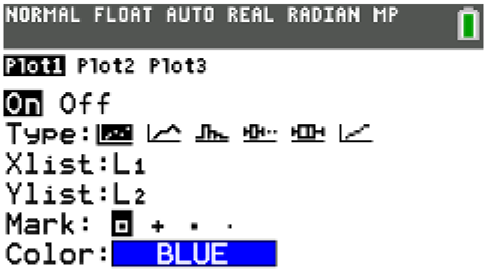
L1	L2	L3	L4	L5
0	0			
1	5			
2	10			
3	15			
4	20			

L2(6)=

3. Create a scatter plot to determine the type of regression equation to use (i.e. linear, quadratic, exponential, sinusoidal). To do this, press 2ND Y=, (below left) then press **1: Plot 1**. This brings you to below right page where you turn the plot "on" by pressing enter on the ON selection. The scatter plot is the first (default) type. The screen then displays as below:

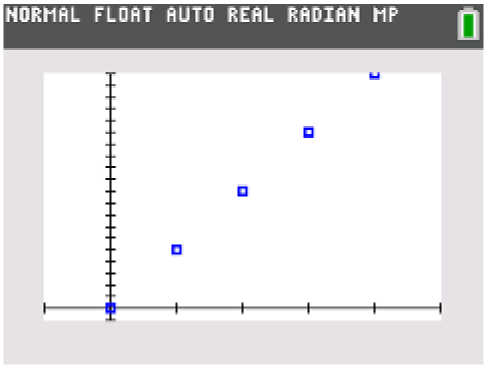


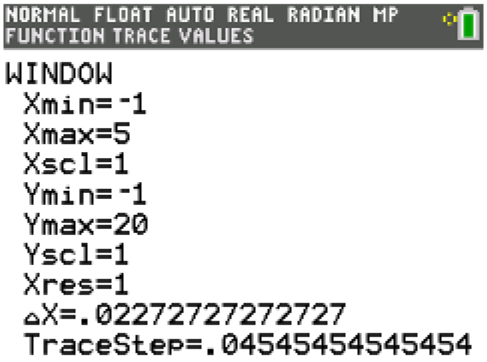
```
NORMAL FLOAT AUTO REAL RADIAN MP
STAT PLOTS
1:Plot1…On
    L1    L2
2:Plot2…Off
    L1    L2
3:Plot3…Off
    L1    L2
4:PlotsOff
5:PlotsOn
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Ylist:L2
Mark: □ + · ·
Color: BLUE
```

4. Press GRAPH to get the graph. You may need to change the window settings to look as given below. See how the plot looks like a line or linear.

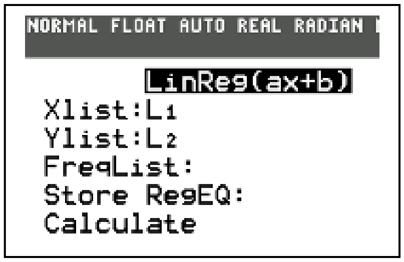




```
NORMAL FLOAT AUTO REAL RADIAN MP
FUNCTION TRACE VALUES
WINDOW
 Xmin=-1
 Xmax=5
 Xscl=1
 Ymin=-1
 Ymax=20
 Yscl=1
 Xres=1
 ΔX=.0227272727272727
 TraceStep=.0454545454545454
```

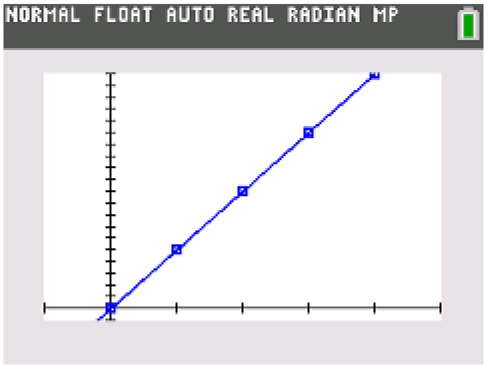
5. Press STAT, then ▶ (the right arrow) to get to the **CALC** menu. The menu below should come up, so choose the type of regression you wish to do. In this example, it is Linear so select **4- LinReg(ax+b)**. Make sure the next screen looks like the right one below, and press ENTER until you move to another screen (5 times – every time the cursor scolls down). This will give you the regression equation of the line/curve of best fit.

- **4: LinReg (ax+b) (Linear Regression)**
- **5: QuadReg (Quadratic Regression)**
- **6: CubicReg (Cubic Regression)**
- **0: ExpReg (Exponential Regression)**
- **9: LnReg (Logarithmic Regression)**
- **C: SinReg (Sinusoidal Regression)**



```
NORMAL FLOAT AUTO REAL RADIAN
       LinReg(ax+b)
 Xlist:L1
 Ylist:L2
 FreqList:
 Store RegEQ:
 Calculate
```

6. Next, you will want to graph the regression equation to see if it goes through most of your points. Press Y=, then VARS, followed by **5: Statistics…** Next, you will press ▶, ▶ to get to **EQ** menu, press **1: RegEq** for the regression equation. That should input the regression equation into the **Y=** menu to graph. So, press GRAPH and see if the line/curve goes through your points!

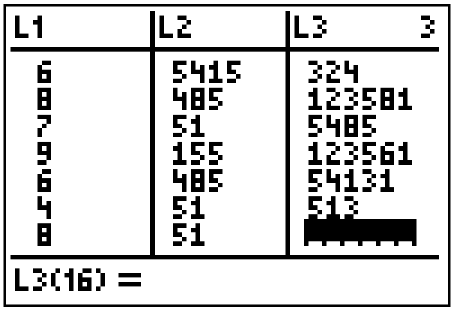


That is how you get the line (equation) of best for your data.

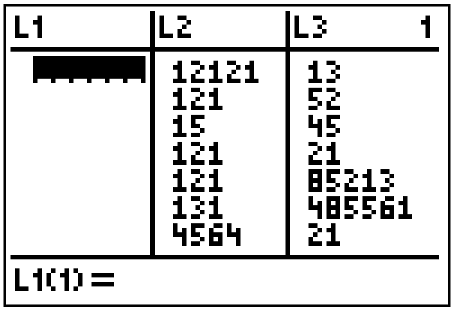
CLEARING THE LISTS:

- **To clear JUST L1.**

1. Highlight the label of the list (L1)
   [STAT] → [ENTER] → [^] → [CLEAR] → [ENTER]



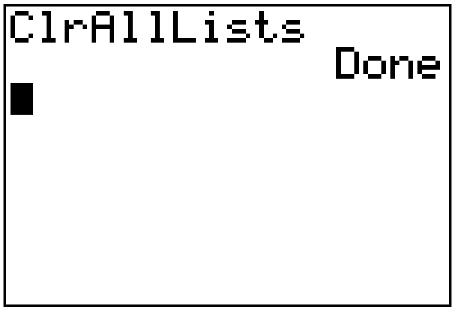
```
L1    |L2    |L3     3
  6   | 5415 | 324
  8   | 485  | 123581
  7   | 51   | 5485
  9   | 155  | 123561
  6   | 485  | 54131
  4   | 51   | 513
  8   | 51   |
L3(16) =
```



```
L1    |L2    |L3     1
      | 12121| 13
      | 121  | 52
      | 15   | 45
      | 121  | 21
      | 121  | 85213
      | 131  | 485561
      | 4564 | 21
L1(1)=
```

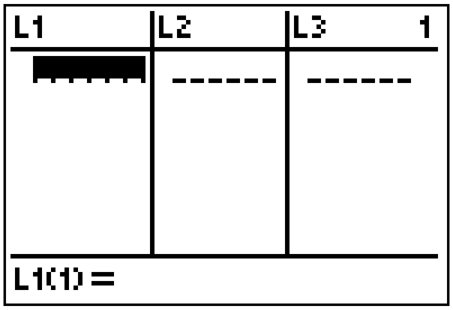
- **To clear all lists**

1. [2ND] → [+] → [4] → [ENTER]



```
ClrAllLists
                Done
```

2. View your cleared lists
   [STAT] → [ENTER]



```
L1    |L2    |L3     1
      |------|------
L1(1)=
```

Graphing a Regression Equation with TI – *nspire* CX

7. Start a new document. Press the

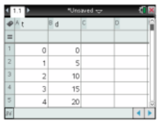
, then choose **1**. When asked to save "Unsaved Document", press **no**.

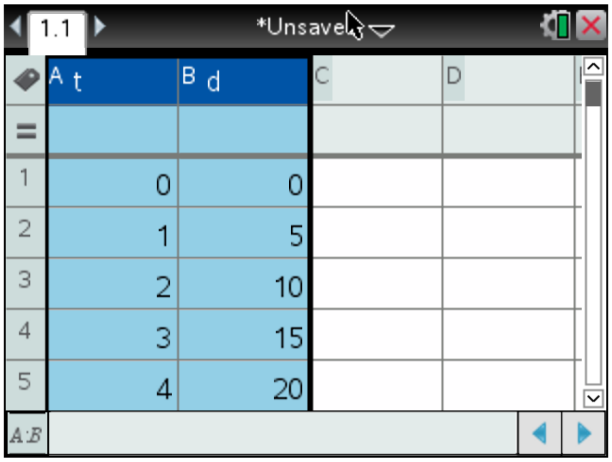
8. When the menu pops up, press **4: Add Lists & Spreadsheet**

9. Put your *x* – label in the box with **A** at the top (for example: *t* for time), press **tab** then put your *y* – label in the box with **B** at the top (for example: *d* for distance).

10. Input your data vertically, as shown.

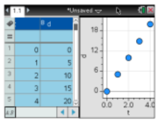
11. Create a scatter plot to determine the type of regression equation to use (i.e. linear, quadratic, exponential, sinusoidal, etc). Highlight columns **A** and **B** by pressing the **up arrow** on column **A** until it turns the entire column blue, then press **shift** and **right** to highlight **B**.

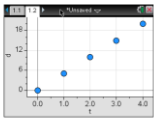




	A t	B d	C	D
=				
1	0	0		
2	1	5		
3	2	10		
4	3	15		
5	4	20		

12. Next, press **Menu**, **3: Data**, **9: Quick Graph**. The graph will appear on the same page as the data, so press **crtl 6** to put it on the next page. Press **crtl right** to look at it and decide what shape it is. In this example, it is Linear.



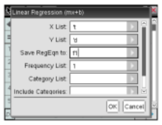


13. Getting the regression equation: **Menu**, **4: Statistics**, **1: Stat Calculations**, then pick your regression:

- **3: Linear Regression (mx+b)**
- **6: Quadratic Regression**

- **7: Cubic Regression**
- **A: Exponential Regression**
- **B: Logarithmic Regression**
- **C: Sinusoidal Regression**

14. Fill in the **X List** and **Y List** by pressing the right arrow at each. The **X List** is the letter in the **A** box and **Y List** is the letter in the **B** box. Then press **tab** until at **OK**, and press **enter**. A list will come up showing the equation and the values to substitute in for *a*, *b*, etc.



Linear Regression (mx+b)

X List: 't

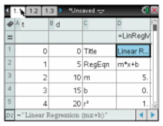
Y List: 'd

Save RegEqn to: f1

Frequency List: 1

Category List:

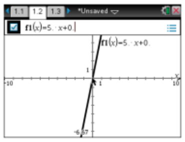
Include Categories:



	A t	B d	C	D
=				=LinRegM
1	0	0	Title	Linear R...
2	1	5	RegEqn	m*x+b
3	2	10	m	5.
4	3	15	b	0.
5	4	20	r²	1.

D1 ="Linear Regression (mx+b)"

15. If you wish to see the graph of the regression equation (to make a table of values, find intercepts, max/min points, etc), press **crtl i** (for insert), then press **2: Add Graphs**. Press the **up** arrow, and the equation should already be typed in for you. You are ready to go!



$f1(x)=5.\cdot x+0.$

Example 1 – Quadratic Regression

The following table shows the tuition for a semester at CBU in dollars for years since 1990.

Year	CBU Tuition
1	3770
3	4440
5	5000
7	5800
9	6570
11	7450
13	8370
15	9315
17	10420
19	11590
21	12760

Let $C(t)$ be the cost of tuition at CBU in dollars for t years since 1990.

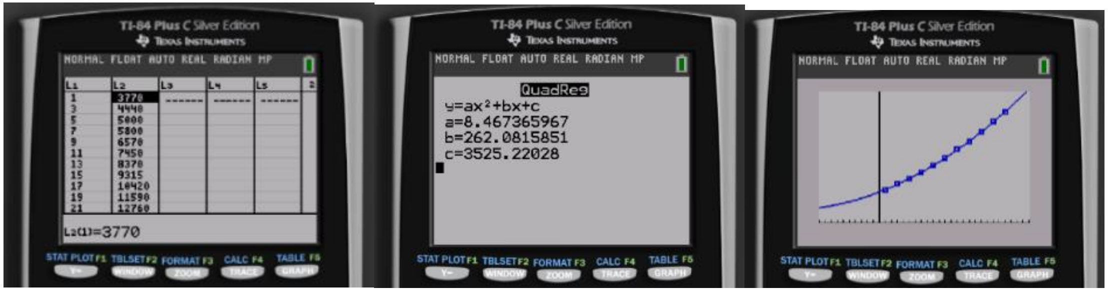
A quadratic model for the data is…

Use three decimals in your answer.

Find the regression. You will need to adjust your window settings.

- ***Your y-max should be larger than your largest y-value in the table. Your x-max should be larger than your largest x-value.***

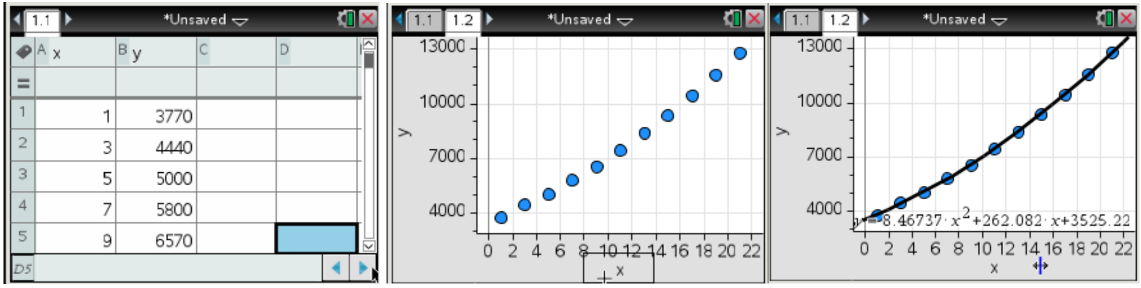
Ti-84


TI-84 Plus C Silver Edition
Texas Instruments
NORMAL FLOAT AUTO REAL RADIAN MP
STAT PLOT F1
TBLSET F2
FORMAT F3
CALC F4
TABLE F5
Y=
WINDOW
ZOOM
TRACE
GRAPH

```
L1   L2     L3      L4      L5
1    3770   ------  ------  ------
3    4440
5    5000
7    5800
9    6570
11   7450
13   8370
15   9315
17   10420
19   11590
21   12760
L2(1)=3770
```

```
QuadReg
y=ax²+bx+c
a=8.467365967
b=262.0815851
c=3525.22028
```

Ti-Nspire


1.1
1.2
*Unsaved
13000
10000
7000
4000
y
0 2 4 6 8 10 12 14 16 18 20 22
x
y=8.46737·x²+262.082·x+3525.22
D5

	A x	B y	C	D
=				
1	1	3770		
2	3	4440		
3	5	5000		
4	7	5800		
5	9	6570		

The regression function is $y = 8.467x^2 + 262.082x + 3525.220$

Example 1 – Continued…

Estimate the tuition in 2016.

The function is for years since 1990, so our x-value will be 2016-1990 = 26. We want to find what y is when x is 26. We sub 26 in for x.

$$y = 8.467x^2 + 262.082x + 3525.220$$

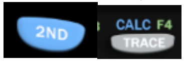
$$y = 8.467(26)^2 + 262.082(26) + 3525.220$$

$$y = \$16063.04$$

Or solve it graphically

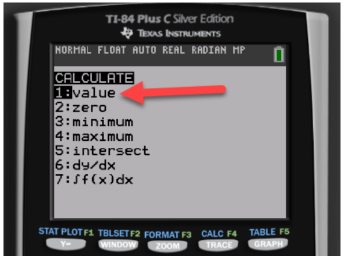
Ti-84

Use the "value" function to determine y when x is 26.

Go to your graph and press 

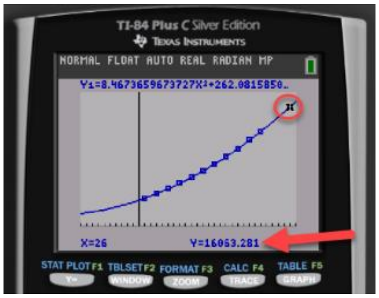
2ND
CALC F4
TRACE

. The select "value"


TI-84 Plus C Silver Edition
Texas Instruments
STAT PLOT F1 Y=
TBLSET F2 WINDOW
FORMAT F3 ZOOM
CALC F4 TRACE
TABLE F5 GRAPH

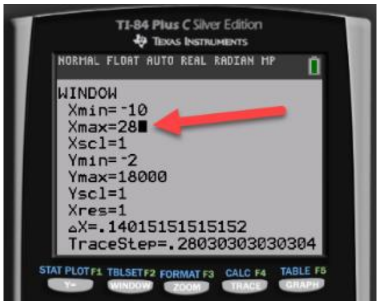
```
NORMAL FLOAT AUTO REAL RADIAN MP
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

Then enter the number 26.


TI-84 Plus C Silver Edition
Texas Instruments
STAT PLOT F1 Y=
TBLSET F2 WINDOW
FORMAT F3 ZOOM
CALC F4 TRACE
TABLE F5 GRAPH

```
NORMAL FLOAT AUTO REAL RADIAN MP
Y1=8.4673659673727X²+262.0815850…
X=26                Y=16063.281
```

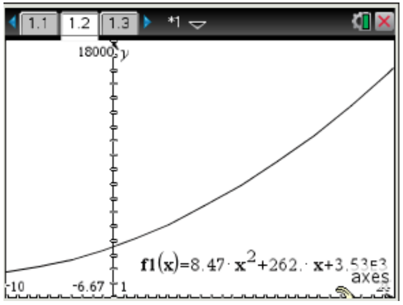
If you get an ***error message***, then you need to change your window settings to an x-value larger than 26


TI-84 Plus C Silver Edition
Texas Instruments
STAT PLOT F1 Y=
TBLSET F2 WINDOW
FORMAT F3 ZOOM
CALC F4 TRACE
TABLE F5 GRAPH

```
NORMAL FLOAT AUTO REAL RADIAN MP
WINDOW
 Xmin=⁻10
 Xmax=28█
 Xscl=1
 Ymin=⁻2
 Ymax=18000
 Yscl=1
 Xres=1
 ΔX=.14015151515152
 TraceStep=.28030303030304
```

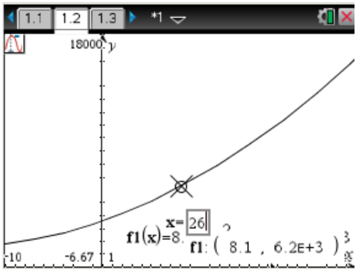
Ti-nspire

Graph the function from above and determine the appropriate window settings.


1.1
1.2
1.3
*1
18000
y
-10
-6.67
1
axes

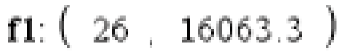
f1(x)=8.47·x²+262.·x+3.53E3

Select Menu→Trace→Trace Graph. Start typing 26. You should see x=26 on the screen.


1.1
1.2
1.3
*1
18000
y
-10
-6.67
1

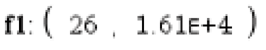
f1(x)=8.
x=26
f1: (8.1 , 6.2E+3)

Press Enter. Your y-value will be your answer.



f1: (26 , 16063.3)

If you get an answer like

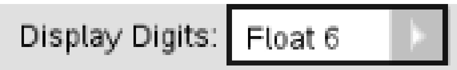


f1: (26 , 1.61E+4)

,

You need to change your settings to show more decimals.

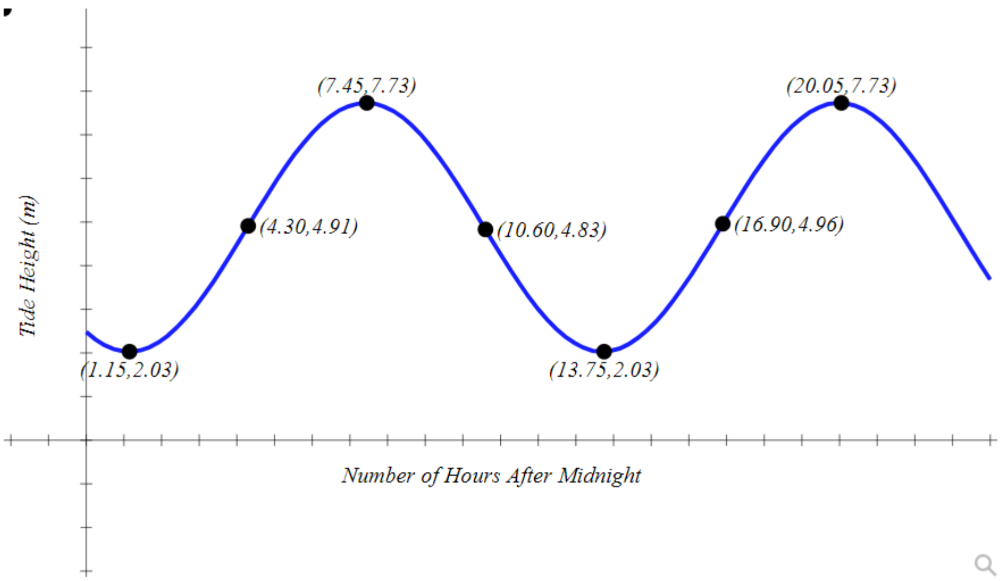
Select **Menu→Settings** Then change display digits to show more than just 3. Make this the Default.



Display Digits: Float 6

Example 2 – Sinusoidal Regression

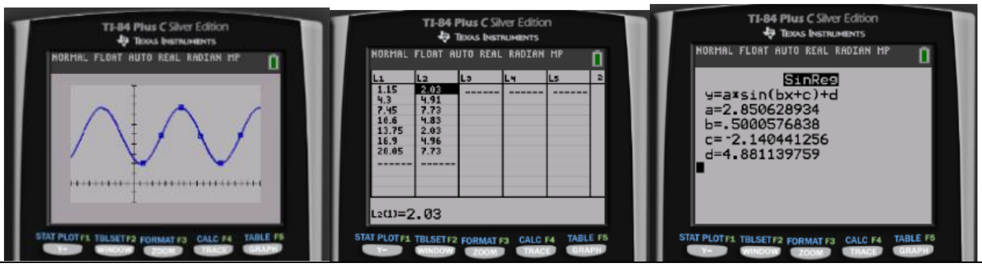
The English Channel is a body of water that separates England and France, and each year swimmers from around the world attempt to swim across it. In preparation for her solo swim, Haley has researched the predicted tide height for the particular day that she plans to swim. The predicted heights of the tide, in metres, as a function of the number of hours after midnight, are shown in the graph below.



State the sinusoidal regression function that represents the graph above in the form $y = a\sin(bx - 2.14) + d$, where y represents the height of the tide, in metres, x hours after midnight. If necessary, round values to the nearest hundredth.

This question might seem tricky because were not given a table. However, we were given points on the curve. Remember that a point gives you an x-value and a y-value. We can turn this into a table of values and find the regression. **Make sure your calculator is set to Radians in document settings**

Ti-84

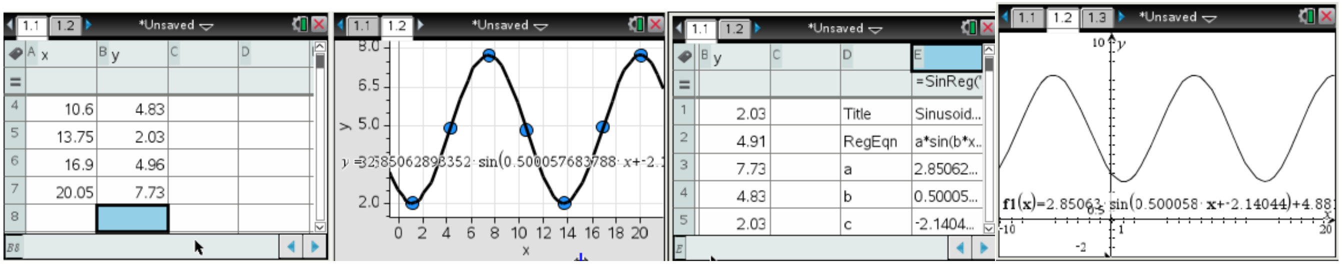

TI-84 Plus C Silver Edition
Texas Instruments
NORMAL FLOAT AUTO REAL RADIAN MP
STAT PLOT F1 TBLSET F2 FORMAT F3 CALC F4 TABLE F5
Y= WINDOW ZOOM TRACE GRAPH

L1	L2	L3	L4	L5
1.15	2.03	------	------	------
4.3	4.91			
7.45	7.73			
10.6	4.83			
13.75	2.03			
16.9	4.96			
20.05	7.73			
------	------			

L2(1)=2.03

```
SinReg
y=a*sin(bx+c)+d
a=2.850628934
b=.5000576838
c=-2.140441256
d=4.881139759
```

Ti-nspire


1.1 1.2 1.3
*Unsaved
8.0
6.5
5.0
2.0
y
0 2 4 6 8 10 12 14 16 18 20
x
10
y
-10
1
20
-2
0.5

	A x	B y	C	D
=				
4	10.6	4.83		
5	13.75	2.03		
6	16.9	4.96		
7	20.05	7.73		
8				

$y = 2.85062893352 \cdot \sin(0.500057683788 \cdot x + -2.1$

	B y	C	D	E
=				=SinReg(
1	2.03		Title	Sinusoid...
2	4.91		RegEqn	a*sin(b*x..
3	7.73		a	2.85062...
4	4.83		b	0.50005...
5	2.03		c	-2.1404...

$f1(x) = 2.85063 \cdot \sin(0.500058 \cdot x + -2.14044) + 4.88$

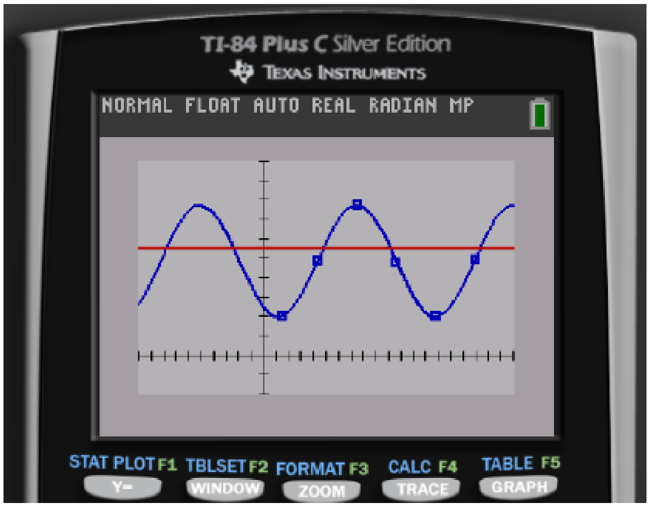
The regression function is $y = 2.85\sin(0.50x - 2.14) + 4.88$

Example 2 – Continued…

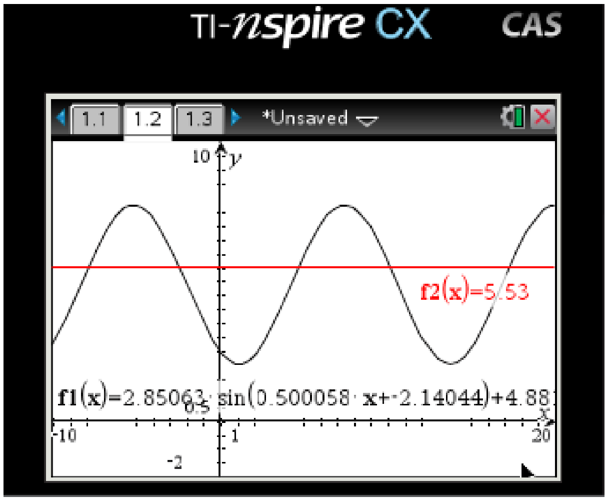
Shannon must start her solo swim at a falling tide of 5.53 m in height. Calculate the fewest number of hours after midnight when Shannon should start her swim, to the nearest hundredth of an hour.

Remember that our y-value is our height, and our x-value is our time. So we were given y and we need to solve for x.

Let's graph the line $y = 5.53$ with our function from above.


TI-84 Plus C Silver Edition
Texas Instruments
STAT PLOT F1
TBLSET F2
FORMAT F3
CALC F4
TABLE F5
Y=
WINDOW
ZOOM
TRACE
GRAPH

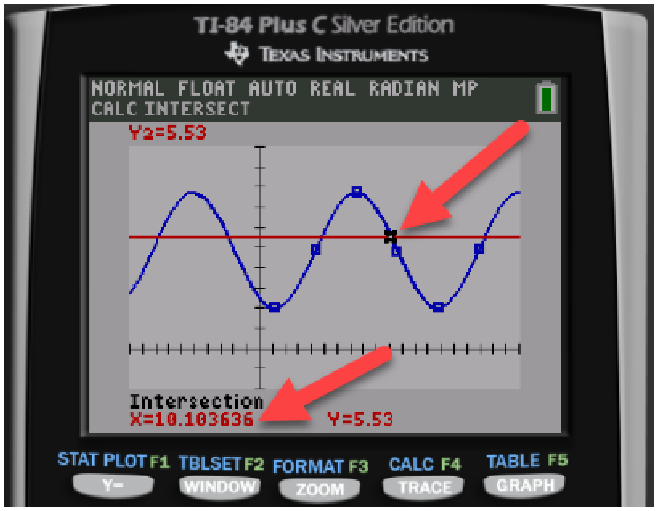
NORMAL FLOAT AUTO REAL RADIAN MP


TI-nspire CX CAS
1.1
1.2
1.3
*Unsaved
10
y
x
-10
1
20
0.5
-2

f2(x)=5.53

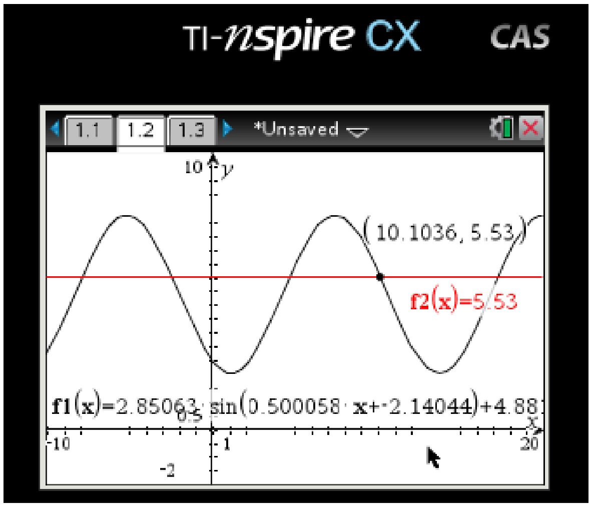
f1(x)=2.85063·sin(0.500058·x+-2.14044)+4.88

We are looking for the ***intersection*** of the two graphs but we need to pick the right one. She must start at a falling tide (going down) and be at a height of 5.53m.


TI-84 Plus C Silver Edition
Texas Instruments
STAT PLOT F1
TBLSET F2
FORMAT F3
CALC F4
TABLE F5
Y=
WINDOW
ZOOM
TRACE
GRAPH

NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y2=5.53

Intersection
X=10.103636 Y=5.53


TI-nspire CX CAS
1.1
1.2
1.3
*Unsaved
10
y
x
-10
1
20
0.5
-2

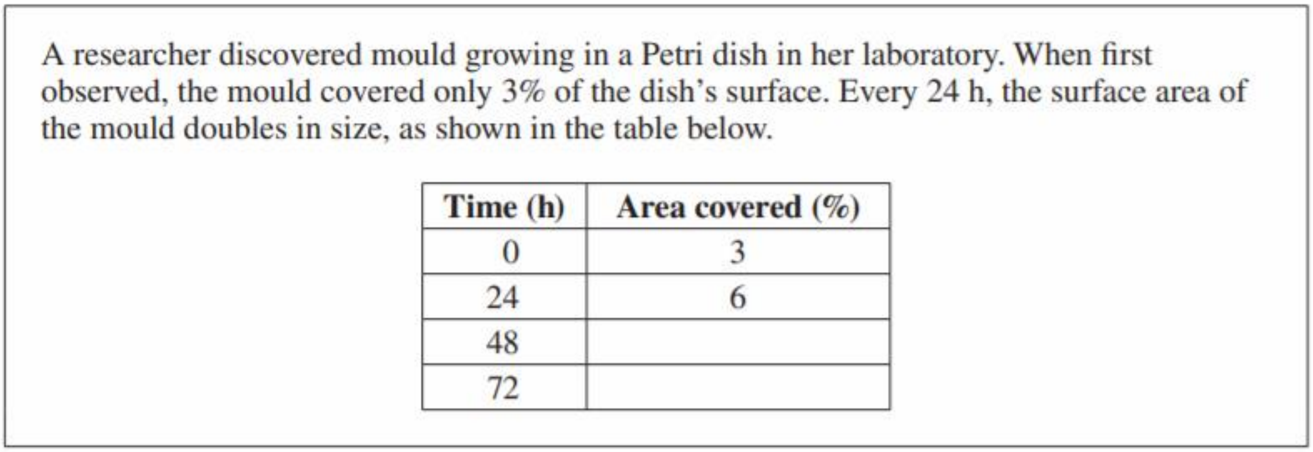
(10.1036, 5.53)

f2(x)=5.53

f1(x)=2.85063·sin(0.500058·x+-2.14044)+4.88

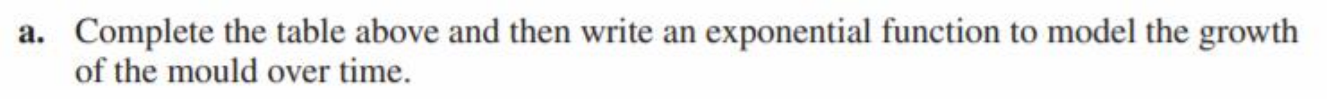
She must start her swim 10.10 hours after midnight!

Example 3– Exponential Regression



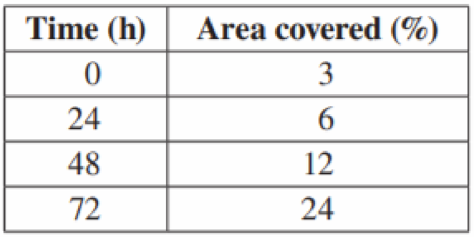
A researcher discovered mould growing in a Petri dish in her laboratory. When first observed, the mould covered only 3% of the dish's surface. Every 24 h, the surface area of the mould doubles in size, as shown in the table below.

Time (h)	Area covered (%)
0	3
24	6
48	
72	



a. Complete the table above and then write an exponential function to model the growth of the mould over time.

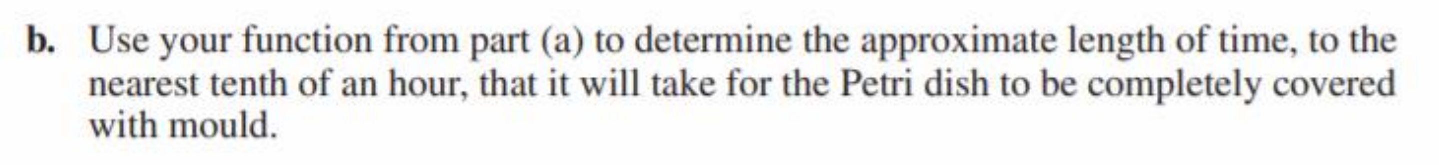
We can recognize from the question that this is an exponential growth problem



Time (h)	Area covered (%)
0	3
24	6
48	12
72	24

After filling in the table, follow the steps to find the regression function for your calculator.

Using regression, $\boldsymbol{y = 3(1.029302237\ldots)^x}$**,** where x represents the number of hours since the first observation and y represents the percentage of area covered.



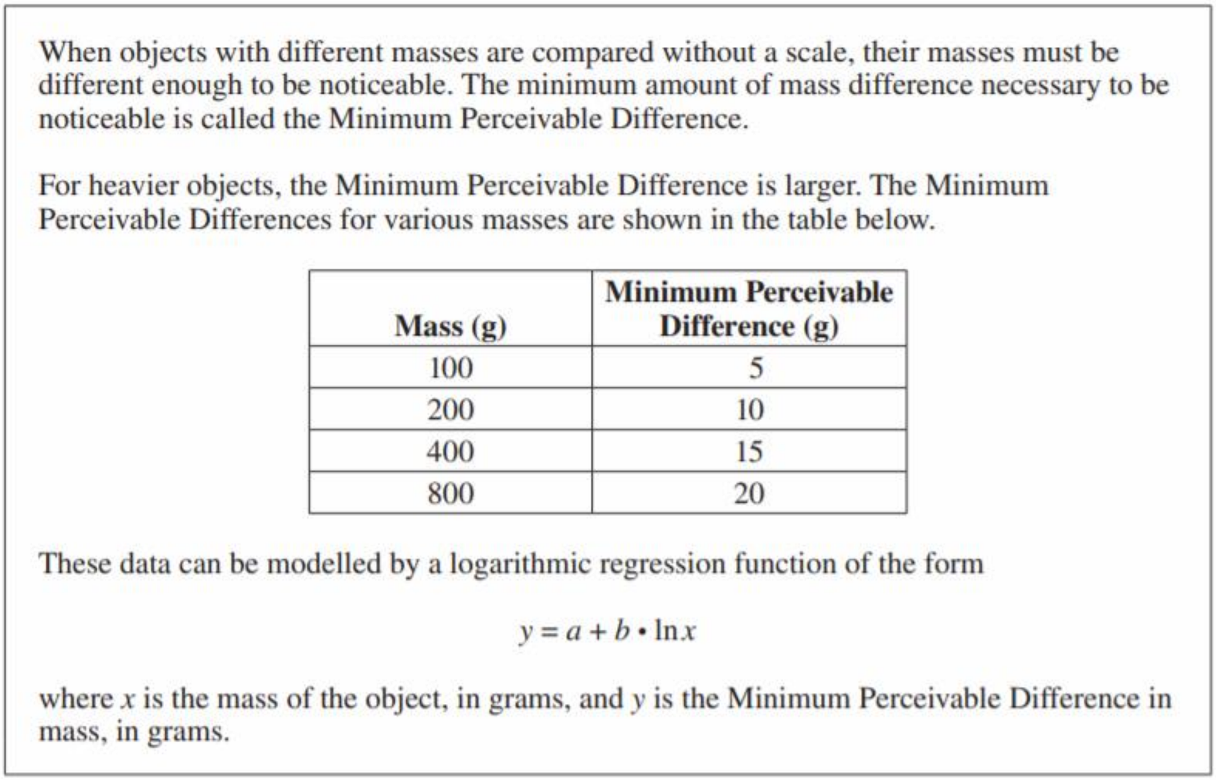
b. Use your function from part (a) to determine the approximate length of time, to the nearest tenth of an hour, that it will take for the Petri dish to be completely covered with mould.

Using your calculator, place $y_1 = 3(1.029302237\ldots)^x$ and $y_2 = 100$

You need to adjust your windows as follows: x: $[0, 150, 10]$ and y: $[0, 110, 10]$

Find the intersection for the two lines, then utilize the x-coordinate. $\boldsymbol{x = 121.4}$

Example 4- Logarithmic Regression



When objects with different masses are compared without a scale, their masses must be different enough to be noticeable. The minimum amount of mass difference necessary to be noticeable is called the Minimum Perceivable Difference.

For heavier objects, the Minimum Perceivable Difference is larger. The Minimum Perceivable Differences for various masses are shown in the table below.

Mass (g)	**Minimum Perceivable Difference (g)**
100	5
200	10
400	15
800	20

These data can be modelled by a logarithmic regression function of the form

$$y = a + b \cdot \ln x$$

where x is the mass of the object, in grams, and y is the Minimum Perceivable Difference in mass, in grams.

a. Determine a logarithmic regression function that could be used to model these data. Round the values of a and b to the nearest tenth.

$$y = -28.2 + 7.2\, ln x$$

b. According to the logarithmic regression function, determine the Minimum Perceivable Difference, to the nearest gram, for an object with a mass of 2100g.

Let $x = 2100$

$y = -28.2 + 7.2 \ln 2100$

$y = 26.96158711$

The Minimum Perceivable Difference in mass would be **approximately 27g.**

***Always watch for the following:**

- Which regression they are asking you to calculate
- To what value do they want you to round to
- Do they give you an x value or a y value?
- Are you looking for an x value or a y value?
- What are your units for your answer?

EX: Using the table below, create a natural logarithmic regression and answer the following questions:

% Carbon-14 (x)	95	79	68	38	27
Age in years (y)	425	1950	3191	8000	10824

a. What is the natural logarithmic regression for the above chart? Round to the nearest integer.

b. A bone fragment with the carved image of a mammoth was discovered in the southern U.S. state of Florida in June 2011. If the carbon dating test indicated that approximately 20.3% of carbon-14 was left, estimate the age of the bone fragment to the nearest 1000 years.

c. If the skeleton of a bison is about 10000 years old, what percentage, to the nearest whole number, of carbon-14 should the paleontologist find remaining in the bones that were found?

EX: A golf club manufacturer, carrying out research on hybrid irons, recorded the path of the golf ball from tee to green on a par 3 hole. The data shows the height, h metres, of the golf ball above the ground after t seconds.

Time (s)	0	0.7	1.4	2.1	2.8	3.5	4.2	4.9
Height (m)	0.03	15.13	25.43	30.92	31.61	27.50	18.59	4.88

a. Create a scatterplot using a graphing calculator to confirm the type of polynomial function that is an appropriate model for the data.

b. Using the appropriate regression feature of the calculator to determine a function that models the data. Round to the nearest hundredth.

c. Determine the height of the golf ball above the ground after 3 seconds.

d. Determine the maximum height of the golf ball above the ground and the time it took for the golf ball to reach the maximum height.

e. Assuming the green is the same height above the ground as the tee box, determine how long it took for the golf ball to hit the ground.

f. Write a suitable domain and range for the polynomial function.

g. Can you explain why the golf ball was $3cm$ above the ground at time $t = 0$?

Homework